PILOT

Final Practices: Solutions

Differential Equations

Summer 2024

1. Solve the following initial value problem, represent your solution as a fundamental matrix:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution:

Here, we first find the eigenvalues for the matrix, that is:

$$\det \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = 0.$$

Therefore, the polynomial is $(1-\lambda)(-7-\lambda)+16 = (\lambda+3)^2 = 0$, hence the eigenvalues is $\lambda_1 = \lambda_2 = -3$. Then, we look for the eigenvectors.

- For $\lambda_1 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $\boldsymbol{\xi}_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} x \\ x - 1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$.

Hence, the general solution is:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \right).$$

By the initial condition, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, so:

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + 0 \\ C_1 - C_2/4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $C_1 = 3$ and $C_2 = 4$, so the particular solution is:

$$\mathbf{x}(t) = \boxed{\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} te^{-3t}}.$$

2.* Let a system of differential equations be defined as follows, find its general solutions:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^3.$$

Solution:

Again, we first find the eigenvalues of the equation, *i.e.*:

$$\det \begin{pmatrix} 1-\lambda & 0 & 4 \\ 1 & 1-\lambda & 3 \\ 0 & 4 & 1-\lambda \end{pmatrix} = 0,$$

which is $(1-\lambda)^3 + 16 - 12(1-\lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda+1)^2(\lambda-5) = 0$.
Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Now, we look for eigenvectors.

- For $\lambda_1 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $x \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$.
- For $\lambda_2 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} 4x \\ x+1 \\ -2x-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
- For $\lambda_3 = 5$, we have $\begin{pmatrix} -4 & 0 & 4 \\ 1 & -4 & 3 \\ 0 & 4 & -4 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$, which is $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + C_2 \left(t e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) + C_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}.$$

3.** Let $\mathrm{Id} \in \mathcal{L}(\mathbb{R}^n)$ be the identity map in an n-dimensional Euclidean space, show that the following equality holds for matrix exponential:

$$\exp(\mathrm{Id}) = e \cdot \mathrm{Id}\,.$$

Hint: Consider the matrix exponential and the Taylor expansion of $\exp(x)$.

Solution:

Proof. Here, we first note that, by definition:

$$\mathrm{Id}^k = \mathrm{Id} \text{ for all } k \in \mathbb{N},$$

thus, we want to expand the matrix exponential as follows:

$$\begin{aligned} \exp(\mathrm{Id}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{Id}^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{Id} \\ &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} \right) \mathrm{Id}\,. \end{aligned}$$

Recall that the Taylor expansion of e^x at 0 is:

$$e^x \sim \sum_{k=0}^{\infty} \frac{1}{k!} e^0 (x-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Evaluating the above equation at 1 gives that:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = e,$$

and hence, we have the matrix exponential as:

$$\exp(\mathrm{Id}) = e \cdot \mathrm{Id},$$

as desired. □

4. Let M be a square matrix, M is defined to be *nilpotent* if $M^k = 0$ for some positive integer k.

   (a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

   Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

   (b)* If all the entries in N are rational, show that $\exp(N)$ has rational entries.

   (c)** Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
   Hint: Use the differences of squares for matrices.

Solution:

(a) *proof of N is nilpotent.* Here, we want to do the matrix multiplication:

$$N^2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we have shown that $N^3 = 0$, or the zero matrix, hence N is nilpotent. □

Then, we want to calculate the matrix exponential, that is:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}.$$

(b) *Proof.* By the definition that N is nilpotent, we know that $N^m = 0$ for some finite positive integer m, hence, we can make the (countable) infinite sum into a finite sum:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \sum_{k=0}^{m} \frac{1}{k!} N^k,$$

thus all the entries are sum and non-zero divisions of rational numbers, while rational numbers are closed under addition and non-zero divisions, hence, all entries of $\exp(N)$ is rational. □

(c) *Proof.* Here, we recall the differences of squares still works when commutativity for multiplications fails, hence the we can still use it for matrix multiplication, namely, for all $m \in \mathbb{Z}^+$:

$$(\mathrm{Id}_n + N) \cdot (\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) = \mathrm{Id}_n - N^{2^{m+1}}$$

Since N is *nilpotent*, this implies that we have some k such that $N^\ell = 0$ for all $\ell \geq k$. Meanwhile, note that $2^\ell \geq \ell$ for all positive integer ℓ. (This can be proven by induction.) Therefore, we select $m + 1 \geq k$ so that $N^{2m+1} = 0$, and we have:

$$(\mathrm{Id}_n + N) \cdot \left[(\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) \right] = \mathrm{Id}_n,$$

thus $\mathrm{Id}_n + N$ is invertible. □

Note that the elements of all n-by-n matrices can be considered as a *ring*, while *nilpotent* can be defined more generally for *rings*. We invite capable readers to investigate more properties of *nilpotent* elements of *rings* in the discipline of *Modern Algebra*.

5. Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a)* Show that $M^\intercal = M^{-1}$.

(b)** Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that $M^n = \mathrm{Id}_2$. Here, n is called the *order* of M.
Hint: Consider the rotational matrix geometrically, rather than arithmetically.

(c)** Let $\theta = \pi/2$, calculate the matrix exponential $\exp(M)$.
Hint: Consider the *order* of M and the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$.

Solution:

(a) *Proof.* Here, we recall the method of inverting a matrix:

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix} \cos\theta & -(-\sin\theta) \\ -\sin\theta & \cos\theta \end{pmatrix} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = M^\intercal.$$

□

(b) Look, we want to analyze this geometrically, if $\theta = 2\pi/k$, then that implies that M is a counter-clockwise rotation of $2\pi/k$, and since a full revolution is 2π, this implies a rotation of k times will make restore to the original vector, *i.e.*, $M^k = \mathrm{Id}_2$. Moreover, for any positive integer less than k, we cannot rotate back to 2π, which implies that the order of M is $\boxed{2}$.

(c) Here, we construct the matrix exponential, note that the order of M is 4, we have:

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{k!}M^k.$$

Here, we want to consider each entry respectively, since each entry is finite and since M has order 4, the absolute value of the sum of the entries must be finite, so each entry converges *absolutely*, hence we are free to change the order of the sum, so we have:

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!}M + \sum_{k=0}^{\infty} \frac{1}{(4k+2)!}M^2 + \sum_{k=0}^{\infty} \frac{1}{(4k+3)!}M^3 + \sum_{k=0}^{\infty} \frac{1}{(4k)!}\,\mathrm{Id}.$$

For the 4 sums of factorials, we note that the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$ at 0 evaluated at $x = 1$ are, respectively:

$$\begin{array}{rclllllll} e^1 & = & +\frac{1}{0!} & +\frac{1}{1!} & +\frac{1}{2!} & +\frac{1}{3!} & +\frac{1}{4!} & +\frac{1}{5!} & +\cdots \\ e^{-1} & = & +\frac{1}{0!} & -\frac{1}{1!} & +\frac{1}{2!} & -\frac{1}{3!} & +\frac{1}{4!} & -\frac{1}{5!} & +\cdots \\ \sin 1 & = & & +\frac{1}{1!} & & -\frac{1}{3!} & & +\frac{1}{5!} & -\cdots \\ \cos 1 & = & +\frac{1}{0!} & & -\frac{1}{2!} & & +\frac{1}{4!} & & -\cdots \end{array}$$

Since the first series converges, we know that the later three series converges *absolutely*, so we are free to move around terms.

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From the expressions, by columns, we can observe that:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} = \frac{e^1 - e^{-1}}{4} + \frac{\sin 1}{2}, \qquad \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} = \frac{e^1 + e^{-1}}{4} - \frac{\cos 1}{2},$$
$$\sum_{k=0}^{\infty} \frac{1}{(4k+4)!} = \frac{e^1 - e^{-1}}{4} - \frac{\sin 1}{2}, \qquad \sum_{k=0}^{\infty} \frac{1}{(4k)!} = \frac{e^1 + e^{-1}}{4} + \frac{\sin 1}{2}.$$

Now, we shall also evaluate the matrices generated by M, that is:

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad M^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$M^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, considering the four entries $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have:

$$a = -\frac{e+1/e}{4} + \frac{\cos 1}{2} + \frac{e+1/e}{4} + \frac{\sin 1}{2} = \frac{\cos 1 + \sin 1}{2},$$
$$b = -\frac{e-1/e}{4} - \frac{\sin 1}{2} + \frac{e-1/e}{4} - \frac{\sin 1}{2} = -2\sin 1,$$
$$c = \frac{e-1/e}{4} + \frac{\sin 1}{2} - \frac{e-1/e}{4} + \frac{\sin 1}{2} = 2\sin 1,$$
$$d = -\frac{e+1/e}{4} + \frac{\cos 1}{2} + \frac{e+1/e}{4} + \frac{\sin 1}{2} = \frac{\cos 1 + \sin 1}{2}.$$

Therefore, the matrix exponential is:

$$\exp(M) = \boxed{\begin{pmatrix} \dfrac{\cos 1 + \sin 1}{2} & -2\sin 1 \\ 2\sin 1 & \dfrac{\cos 1 + \sin 1}{2} \end{pmatrix}}.$$

In particular, mathematicians has considered the *rotation* and *flipping* of regular polygons as the *dihedral groups*, where symmetries and combinatorics play an important role. Please think of ways you may "manipulate" a polygon such that the polygon looks the same.

6. Let a non-linear system be:

$$\frac{dx}{dt} = x - y^2 \text{ and } \frac{dy}{dt} = x + x^2 - 2y.$$

Verify that $(0,0)$ is a critical point and classify its type and stability.

Solution:

proof that $(0,0)$ *is critical point.* The verification of $(0,0)$ being a critical point is trivial. We check that dx/dt and dy/dt evaluated at $(0,0)$ are:

$$\left.\frac{dx}{dt}\right|_{(0,0)} = 0 \text{ and } \left.\frac{dy}{dt}\right|_{(0,0)} = 0,$$

and hence $(0,0)$ is a critical point. □

In particular, denoting $\mathbf{x} = (x, y)$, we verify the linear approximation as:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$

and we note that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, and by:

$$\lambda_2 < 0 < \lambda_1,$$

we know that we have a $\boxed{\text{unstable saddle point}}$ at $(0,0)$.

7. Let a system of non-linear differential equations be defined as follows:

$$\begin{cases} x' = 2x + 3y^2, \\ y' = x + 4y^2. \end{cases}$$

Find all equilibrium(s) and classify their stability locally.

Solution:
Here, we note that the equilibrium(s) is achieved if and only if $x' = y' = 0$, that is:

$$\begin{cases} 2x + 3y^2 = 0, \\ x + 4y^2 = 0. \end{cases}$$

In particular, we consider $z = y^2$, so we have a system of linear equations, that is:

$$\begin{cases} 2x + 3z = 0, \\ x + 4z = 0. \end{cases}$$

Meanwhile, the above system simplifies to $x = y = 0$, hence the only equilibrium is at $(x, y) = (0, 0)$. Then, we consider the system locally, denoting $\mathbf{x} = (x, y)$, that is:

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x},$$

where the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0$. Note that one eigenvalue is zero and the other is positive, then the critical point is $\boxed{\text{unstable}}$.

8. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:

$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$

Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

(a) Find the set of all equilibrium(s) for $\mathbf{x}$.

(b) Find the set in which the equilibrium(s) is locally linear.

Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.

(c)** Construct a function F such that $\mathbf{x}$ has an equilibrium that is <u>not</u> locally linear.
Hint: Consider the condition in which a non-linear system is locally linear.

Solution:

(a) Here, we note that the equilibrium is when $F(\mathbf{x}) = 0$, *i.e.*, $\sin x_1 + \csc(3x_2) = 0$. Here, we note that the image of $\sin x_1$ is $[-1, 1]$ and the image of $\sec(3x_2)$ is $(-\infty, -1] \sqcup [1, \infty)$, this implies that $\sin x_1 + \sec(3x_2)$ is zero only if $\sin x_1 = \pm 1$ and $\sec(3x_2) = \mp 1$, correspondingly.
First, we consider the set in which x_1 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Correspondingly, we consider the set in which x_2 is -1, that is:

$$\left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Then, we consider the set in which x_1 is -1, that is:

$$\left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Likewise, we consider the set in which x_2 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Therefore, set theoretically, we have the set of all equilibriums as:

$$\boxed{\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}}.$$

(b) Note that $\sin x_1$ is (twice) differentiable over the entire domain $\mathbb{R}$ and $\csc(3x_2)$ is (twice) differentiable on all neighborhoods when $\csc(3x_2)$ is ∓ 1, hence the partial derivatives of $F(\mathbf{x})$ with respect to x_1 or x_2 are (twice) differentiable on the neighborhood on all equilibriums, hence the set in which the equilibrium(s) is locally linearly is the same from part (a), namely:

$$\boxed{\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}}.$$

(c) Clearly, we must enforce that $F(\mathbf{x})$ is $\boxed{\text{not twice differentiable}}$ with some partial derivatives near the equilibrium point(s). One trivial example could be using the absolute value, such as $F(\mathbf{x}) = |x_1| + |x_2|$, where $(0, 0)$ is a equilibrium but it is <u>not differentiable</u>.
For capable readers, we invite them to look for more functions, such as the Weierstrass Function, a continuous function that is *nowhere* differentiable:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cos(3^k x).$$

9. Let a system of (x, y) be functions of variable t, and they have the following relationship:
$$x' = (1+x)\sin y \text{ and } y' = 1 - x - \cos y.$$

   (a) Identify the corresponding linear system.

   (b) Evaluate the stability for the equilibrium at $(0,0)$ by showing it is locally linear.

Solution:

(a) Here, since we can write:
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y \\ 1 - \cos y \end{pmatrix},$$
this implies that the linear system is:
$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

(b) $(0,0)$ *is locally linear.* We find the Jacobian Matrix, that is:
$$\mathbf{J} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix}.$$
As we evaluate $\mathbf{J}$ at $(0,0)$ and take its determinant, we have:
$$\det\left(\mathbf{J}\big|_{(0,0)}\right) = \det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 \neq 0.$$
Hence, the $(0,0)$ is locally linear. □

Note that we have found the linear system in part (a), whose eigenvalues are $\lambda_1 = \lambda_2 = 0$. Since $x' = 0$, it indicates that x is a constant, whereas for $y' = -x$ indicates that it will be a $\boxed{\text{unstable almost everywhere}}$ for all neighborhoods of $(0,0)$.
In particular, readers could illustrate the "slope field" for the linear system in (a), and they should notice that except for $x = 0$ being entirely stable, all other trajectory would shift vertically at a constant rate. However, the line $x = 0$ will always be insignificant enough (having *Lebesgue measure* 0), hence we claim that it is unstable almost everywhere. For interested readers, please explore *Lebesgue measure* as a way to determine how large a subset is in Euclidean space.

10.** Let a locally linearly system be defined as:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x} + \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. Find the necessary condition(s) in which the equilibrium(s) have a stable *center* in linear system. Then, state the stability and type (if possible).
Hint: Consider the solution for the linear case or matrix exponential.

Solution:
Without loss of generality, we assume that the system of $\mathbf{x}$ has equilibrium(s), else the statement is vacuously true. Now, we start to evaluate the additional conditions with such assumption:

(i) Note that the system needs to be locally linearly, *i.e.*, we must have $\boxed{\mathbf{f}(\mathbf{x}) \text{ being twice differentiable with respect to partial derivatives}}$.

(ii) Moreover, we need to worry about the linear system to have a *stable center*, that is:

$$\mathbf{x}' = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x}.$$

Note that the eigenvalues would be the solutions to $(\lambda - r)^2 + \mu^2 = 0$, that is $r = \lambda \pm i\mu$, which is a pair of complex conjugates. Here, in to be stable, we want $\lambda \leq 0$, and for center, this forces $\boxed{\lambda = 0}$.

Note that even the linear system is a stable center, the stability of the non-linear system is $\boxed{\text{indeterminate}}$, and the type is $\boxed{\text{center or spiral point}}$.