Final Practices

Differential Equations

Summer 2024

Instructions:

The set of questions serves as PILOT practices to final for the Summer 2024 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- Note that the final is cumulative, while this practice set only contains materials after the second midterm. Please refer to Midterm 1 and 2 Practices for materials covered in the first two midterms.
- Questions marked with asterisk(s) are more challenging, but still accessible, compared to the others.
- The questions are designed to be done without notes and calculators.
- Solutions to the questions will be discussed during the PILOT review session.

1. Solve the following initial value problem, represent your solution as a fundamental matrix:
$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

2.* Let a system of differential equations be defined as follows, find its general solutions:
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^3.$$

3.** Let $\mathrm{Id} \in \mathcal{L}(\mathbb{R}^n)$ be the identity map in an n-dimensional Euclidean space, show that the following equality holds for matrix exponential:
$$\exp(\mathrm{Id}) = e \cdot \mathrm{Id}\,.$$
Hint: Consider the matrix exponential and the Taylor expansion of $\exp(x)$.

4. Let M be a square matrix, M is defined to be *nilpotent* if $M^k = 0$ for some positive integer k.

(a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

(b)* If all the entries in N are rational, show that $\exp(N)$ has rational entries.

(c)** Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
Hint: Use the differences of squares for matrices.

5. Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:
$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
   (a)* Show that $M^\intercal = M^{-1}$.

   (b)** Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that $M^n = \mathrm{Id}_2$. Here, n is called the *order* of M.
   Hint: Consider the rotational matrix geometrically, rather than arithmetically.

   (c)** Let $\theta = \pi/2$, calculate the matrix exponential $\exp(M)$.
   Hint: Consider the *order* of M and the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$.

6. Let a non-linear system be:
$$\frac{dx}{dt} = x - y^2 \text{ and } \frac{dy}{dt} = x + x^2 - 2y.$$
Verify that $(0,0)$ is a critical point and classify its type and stability.

7. Let a system of non-linear differential equations be defined as follows:
$$\begin{cases} x' = 2x + 3y^2, \\ y' = x + 4y^2. \end{cases}$$
Find all equilibrium(s) and classify their stability locally.

8. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:
$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$
Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

   (a) Find the set of all equilibrium(s) for $\mathbf{x}$.

   (b) Find the set in which the equilibrium(s) is locally linear.

   Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.

   (c)** Construct a function F such that $\mathbf{x}$ has a equilibrium that is <u>not</u> locally linear.
   Hint: Consider the condition in which a non-linear system is locally linear.

9. Let a system of (x, y) be functions of variable t, and they have the following relationship:
$$x' = (1 + x)\sin y \text{ and } y' = 1 - x - \cos y.$$
   (a) Identify the corresponding linear system.

   (b) Evaluate the stability for the equilibrium at $(0,0)$ by showing it is locally linear.

10.** Let a locally linearly system be defined as:
$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x} + \mathbf{f}(\mathbf{x}),$$
where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. Find the necessary condition(s) in which the equilibrium(s) have a stable *center* in linear system. Then, state the stability and type (if possible).
Hint: Consider the solution for the linear case or matrix exponential.