Midterm 1 Practices: Solutions

Differential Equations

Summer 2024

1. Find the general solution for $y = y(t)$:
$$y' + 3y = t + e^{-2t},$$
then, describe the behavior of the solution as $t \to \infty$.

Solution:
Here, one could note that this differential equation is not separable but in the form of integrating factor problem, then we find the integrating factor as:
$$\mu(t) = \exp\left(\int_0^t 3ds\right) = \exp(3t).$$
By multiplying both sides with $\exp(3t)$, we obtain the equation:
$$y'e^{3t} + 3ye^{3t} = te^{3t} + e^{-2t}e^{3t}.$$
Clearly, we observe that the left hand side is the derivative after product rule for ye^{3t} and the right hand side can be simplified as:
$$\frac{d}{dt}[ye^{3t}] = te^{3t} + e^t.$$
Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:
$$\begin{aligned} ye^{3t} &= \int te^{3t}dt + \int e^t dt \\ &= \frac{te^{3t}}{3} - \int \frac{1}{3}e^{3t}dt + e^t + C \\ &= \frac{te^{3t}}{3} - \frac{e^{3t}}{9} + e^t + C. \end{aligned}$$
Eventually, we divide both sides by e^{3t} to obtain that:
$$y(t) = \boxed{\frac{t}{3} - \frac{1}{9} + e^{-2t} + Ce^{-3t}}.$$

2. Given an initial value problem:

$$\begin{cases} \dfrac{dy}{dt} - \dfrac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$

(a) Find the integrating factor $\mu(t)$.

(b) Solve for the particular solution for the initial value problem.

(c) Discuss the behavior of the solution as $t \to \infty$ for different cases of y_0.

Solution:

(a) As instructed, we look for the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t -\frac{3}{2}ds\right) = \boxed{\exp\left(-\frac{3}{2}t\right)}.$$

(b) With the integrating factor, we multiply both sides by $\mu(t)$ to obtain that:

$$y'e^{-3t/2} - \frac{3}{2}ye^{-3t/2} = 3te^{-3t/2} + 2e^te^{-3t/2}.$$

Clearly, we observe that the left hand side is the derivative after product rule for $ye^{-3t/2}$ and the right hand side can be simplified as:

$$\frac{d}{dt}\left[ye^{-3t/2}\right] = 3te^{-3t/2} + 2e^{-t/2}.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{-3t/2} &= \int 3te^{-3t/2}dt + \int 2e^{-t/2}dt \\ &= -2te^{-3t/2} + 2\int e^{-3t/2}dt - 4r^{-t/2} + C \\ &= -2te^{-3t/2} - \frac{4}{3}e^{-3t/2} - 4r^{-t/2} + C. \end{aligned}$$

Then, we divide both sides by $e^{-3t/2}$ to get the general solution:

$$y(t) = -2t - \frac{4}{3} - 4e^t + Ce^{3t/2}.$$

Given the initial condition, we have that:

$$y_0 = 0 - \frac{4}{3} - 4 + C,$$

which implies $C = 16/3 + y_0$, leading to the particular solution that:

$$y(t) = \boxed{-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}}.$$

(c) We observe that:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}\right].$$

Note that the important terms are e^t and $e^{3t/2}$, we need to care the critical value $-16/3$:

- when $y_0 > -16/3$, $\boxed{y(t) \to \infty \text{ when } t \to \infty}$,
- when $y_0 \leq -16/3$, $\boxed{y(t) \to -\infty \text{ when } t \to \infty}$.

3. Suppose $f(x)$ is non-zero, let an initial value problem be:
$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

   (a) Show that the differential equation is **not** linear.

   For the next two questions, suppose $f(x) = \tan x$.

   (b) State, without justification, the open interval(s) in which $f(x)$ is continuous.

   (c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

   Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

   (d)** Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

Solution:

(a) *Proof.* We can write the equation as:
$$F(x, y, y') := y' - \frac{xf(x)}{(y+1)(y-1)} = 0,$$
Note that:
$$F(x, (y+1), (y+1)') = y' - \frac{xf(x)}{(y+2)y} \neq 1,$$
so the function is non-linear. □

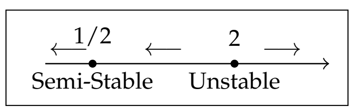
(b) Here, we should consider that:
$$f(x) = \tan x = \frac{\sin x}{\cos x},$$
so the discontinuities are at when $\cos x = 0$, that is:
$$x \in \left\{ \frac{(2k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$
Hence, we have the intervals in which $f(x)$ being continuous as:
$$\boxed{\left\{ \left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2} \right) : k \in \mathbb{Z} \right\}}.$$

(c) *Proof.* Here, we want to write our equation in the standard form and obtain that:
$$y' := f(t, y) = \frac{x \tan x}{(y+1)(y-1)},$$
$$\frac{\partial f(t, y)}{\partial y} = -\frac{x \tan x \cdot 2y}{(y^2-1)^2}.$$
Clear, we note the discontinuities of y at $y = \pm 1$, and x demonstrated as above, thus we can form a rectangle $Q = (-\pi/2, \pi/2) \times (-1, 1)$ in which the initial condition $(0, 0) \in Q$ and $f(t, y)$ with $\partial_y f(t, y)$ are continuous on the interval. By the *existence and uniqueness theorem for non-linear case*, we know that there exists some δ such that there is a unique solution for $-\delta < x < \delta$. □

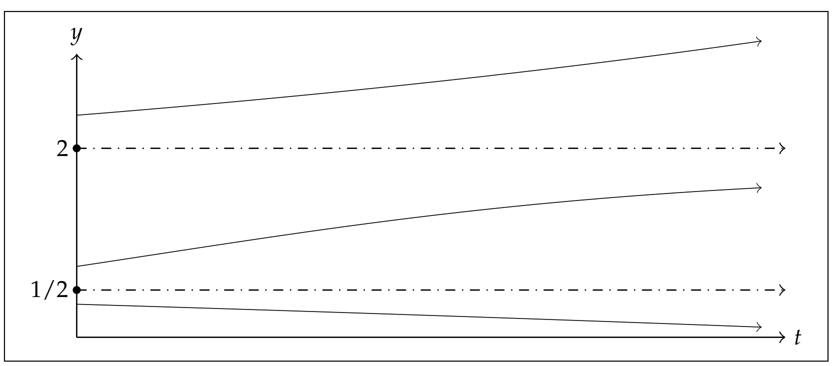
(d) If the condition in (c) does not hold, by contraposition, this implies that continuity must fail, *i.e.*, $\boxed{xf(x) \text{ must be discontinuous at } x = 0}$. Hence, some examples could be:
$$f(x) = \frac{1}{x^2}, \text{ or } \log x, \text{ or } \csc x, \text{ or } \chi_{\{0\}}(x) \text{ etc.}$$

4. An autonomous differential equation is given as follows:
$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \text{ where } t \geq 0 \text{ and } y \geq 0.$$
Draw a phase portrait and sketch a few solutions with different initial conditions.

Solution:
Recall from Pre-Calculus (or Algebra) the following *Rational root test*:
Theorem 4.1: Rational Root Test. Let the polynomial:
$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$
have integer coefficients $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$, then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p,q) = 1$ satisfies that $p|a_0$ and $q|a_n$. ⌟
From the theorem, we can note that if the equation has a rational root, it must be one of:
$$r = \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}.$$
By plugging in, one should notice that $y = 2$ is a root (one might also notice $1/2$ is a root as well, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:
$$\frac{4y^3 - 12y^2 + 9y - 2}{y-2} = 4y^2 - 4y + 1.$$
Clear, we can notice that the right hand side is a perfect square (else, you could use the quadratic formula) that:
$$4y^2 - 4y + 1 = (2y-1)^2.$$
Thus, we now know that the roots are 2 and $1/2$ (multiplicity 2). Hence, the phase portrait is:



Correspondingly, we can sketch a few solutions (not necessarily in scale):



Note that for the **Theorem 4.1**, it can also be generalized into the following manner (in ring theory):
Theorem 4.2: Rational Root Theorem. Let R be UFD, and polynomial:
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in R[x],$$
and let $r = p/q \in K(R)$ be a root of f with $p, q \in R$ and $\gcd(p,q) = 1$, then $p|a_0$ and $q|a_n$. ⌟
The proofs of **Theorem 4.1** and/or **Theorem 4.2** are left as exercises to diligent readers. *Moreover, capable readers should attempt to prove that a polynomial of degree 3 with integer coefficients must have at least one rational root.*

5.* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:

$$y'(x) = e^{2x} + y(x) - 1.$$

Solution:

First, we write the equation in the general form:

$$\frac{dy}{dx} + (1 - e^{2x} - y) = 0.$$

Now, we take the partial derivatives to obtain that:

$$\frac{\partial}{\partial y}[1 - e^{2x} - y] = -1,$$

$$\frac{\partial}{\partial x}[1] = 0.$$

Notice that the mixed partials are not the same, the equation is $\boxed{\text{not exact}}$.

Here, we choose the integrating factor as:

$$\begin{aligned}\mu(x) &= \exp\left(\int_0^x \frac{\frac{\partial}{\partial y}[1 - e^{2s} - y] - \frac{\partial}{\partial s}[1]}{1} ds\right)\\ &= \exp\left(\int_0^x -ds\right) = \exp(-x).\end{aligned}$$

Therefore, our equation becomes:

$$(e^{-x})\frac{dy}{dx} + (e^{-x} - e^{x} - ye^{-x}) = 0.$$

After multiplying the integrating factor, it would be exact. *We leave the repetitive check as an exercise to the readers*.

Now, we can integrate to find the solution with a $h(y)$ as function:

$$\varphi(x, y) = \int (e^{-x} - e^{x} - ye^{-x})dx = -e^{-x} - e^{x} + ye^{-x} + h(y).$$

By taking the partial derivative with respect to y, we have:

$$\partial_y \varphi(x, y) = e^{-x} + h'(y),$$

which leads to the conclusion that $h'(y) = 0$ so $h(y) = C$.

Then, we can conclude that the solution is now:

$$\varphi(x, y) = -e^{-x} - e^{x} + ye^{-x} + C = 0,$$

which is equivalently:

$$y(x) = \boxed{\widetilde{C}e^{x} + 1 + e^{2x}}.$$

6. Let a differential equation be defined as:
$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$
Use Euler's Method with step size $h = 1$ to approximate $y(5)$.

Solution:
With $y(0) = 0$, we have $y'(0) = 0 - 0 = 0$. We do the following steps:

- We approximate $y(1) \approx y(0) + 1 \cdot y'(0) = 0$, then we have $y'(1) \approx 1 - 0 = 1$.
- We approximate $y(2) \approx y(1) + 1 \cdot y'(1) \approx 1$, then we have $y'(2) \approx 2 - 1 = 1$.
- We approximate $y(3) \approx y(2) + 1 \cdot y'(2) \approx 2$, then we have $y'(3) \approx 3 - 2 = 1$.
- We approximate $y(4) \approx y(3) + 1 \cdot y'(3) \approx 3$, then we have $y'(4) \approx 4 - 3 = 1$.
- We approximate $y(5) \approx y(4) + 1 \cdot y'(4) \approx 4$.

Then, we have approximated that:
$$y(5) \approx \boxed{4}.$$

7. Solve the following second order differential equations for $y = y(x)$:

(a) $$y'' + y' - 132y = 0.$$

(b) $$y'' - 4y' = -4y.$$

(c) $$y'' - 2y' + 3y = 0.$$

Solution:

(a) We find the characteristic polynomial as $r^2 + r - 132 = 0$, which can be trivially factorized into:

$$(r - 11)(r + 12) = 0,$$

so with roots $r_1 = 11$ and $r_2 = -12$, we have the general solution as:

$$y(x) = \boxed{C_1 e^{11x} + C_2 e^{-12x}}.$$

(b) We turn the equation to the standard form:

$$y'' - 4y' + 4 = 0.$$

We find the characteristic polynomial as $r^2 - 4r + 4 = 0$, which can be immediately factorized into:

$$(r - 2)^2 = 0,$$

so with roots $r_1 = r_2 = 2$ (repeated roots), we have the general solution as:

$$y(x) = \boxed{C_1 e^{2x} + C_2 x e^{2x}}.$$

(c) We find the characteristic polynomial as $r^2 - 2r + 3 = 0$, which the quadratic formula gives:

$$r = \frac{2 \pm \sqrt{2^2 - 4 \times 3}}{2} = 1 \pm \mathrm{i}\sqrt{2}$$

so with roots $r_1 = 1 + \mathrm{i}\sqrt{2}$ and $r_2 = 1 - \mathrm{i}\sqrt{2}$, we would have the solution:

$$y(x) = C_1 e^{(1+\mathrm{i}\sqrt{2})x} + C_2 e^{(1-\mathrm{i}\sqrt{2})x}.$$

To obtain real solution, we apply Euler's identity:

$$y_1(x) = e^x \big(\cos(\sqrt{2}x) - \mathrm{i}\sin(\sqrt{2}x)\big) \text{ and } y_2(x) = e^x \big(\cos(-\sqrt{2}x) - \mathrm{i}\sin(-\sqrt{2}x)\big).$$

By the *principle of superposition*, we can linearly combine the solutions to be different solutions, so we have:

$$\widetilde{y_1}(x) = \frac{1}{2}(y_1 + y_2) = e^x \cos(\sqrt{2}x),$$

$$\widetilde{y_2}(x) = \frac{1}{2}(y_2 - y_1) = e^x \sin(\sqrt{2}x).$$

One can verify that $\widetilde{y_1}$ and $\widetilde{y_2}$ are linearly independent by taking Wronskian,*i.e.*:

$$\mathrm{W}[\widetilde{y_1}, \widetilde{y_2}] = \det \begin{pmatrix} e^x \cos(\sqrt{2}x) & e^x \sin(\sqrt{2}x) \\ e^x \cos(\sqrt{2}x) - \sqrt{2}e^x \sin(\sqrt{2}x) & e^x \sin(\sqrt{2}x) + \sqrt{2}e^x \cos(\sqrt{2}x) \end{pmatrix}$$

$$= \sqrt{2}e^{2x} \cos^2(\sqrt{2}x) + \sqrt{2}e^{2x} \sin^2(\sqrt{2}x) = \sqrt{2}e^{2x} \neq 0.$$

Now, they are linearly independent, so we have the general solution as:

$$y(x) = \boxed{C_1 e^x \cos(\sqrt{2}x) + C_2 e^x \sin(\sqrt{2}x)}.$$

8.** The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x,t)$:

$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x,0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x,0) = \sin\left(\dfrac{5\pi x}{L}\right), & \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0,t) = u(L,t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$

where c is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x,t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is non-trivial.
Hint: Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis.

Solution:
With the method of separation, we insert the separations back to the system of equation to obtain:

$$v(x)w''(t) = c^2 v''(x) w(t).$$

Now, we apply the separation and set the common ratio to be λ:

$$\frac{v''(x)}{v(x)} = \frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = \lambda.$$

Reformatting the boundary condition gives use the following initial value problem:

$$\begin{cases} v''(x) - \lambda v(x) = 0, \\ v(0) = v(L) = 0. \end{cases}$$

As a second order linear ordinary differential equation, we discuss all following cases:

- If $\lambda = 0$, then $v(x) = a + Bx$ and by the initial condition, $A = B = 0$, which gives the trivial solution, *i.e.*, $v(x) = 0$;
- If $\lambda = \mu^2 > 0$, then we have $v(x) = Ae^{-\mu x} + Be^{\mu x}$ and again giving that $A = B = 0$, or the trivial solution;
- Eventually, if $\lambda = -\mu^2 < 0$, then we have the solution as:

  $$v(x) = A\sin(\mu x) + B\cos(\mu x),$$

  and the initial conditions gives us that:

  $$\begin{cases} v(0) = B = 0, \\ v(L) = A\sin(\mu L) + B\cos(\mu L) = 0, \end{cases}$$

  where A is arbitrary, $B = 0$, and $\mu L = m\pi$ positive integer m.

Overall, the only non-trivial solution would be:

$$v_m(x) = A\sin(\mu_m x), \text{ where } \mu_m = \frac{m\pi}{L}.$$

Eventually, by inserting back $\lambda = -\mu_m^2$, we have $\lambda = -m^2\pi^2/L^2$, giving the solution to $w_m(t)$, another second order linear ordinary differential equation, as:

$$w_m(t) = C\cos(\mu_m ct) + D\sin(\mu_m ct), \text{ with } C, D \in \mathbb{R}.$$

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By the *principle of superposition*, we can have our solution in the form:

$$u(x,t) = \sum_{m=1}^{\infty} [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin(\mu_m x),$$

where our coefficients a_m and b_m have to be chosen to satisfy the initial conditions for $x \in [0, L]$:

$$u(x,0) = \sum_{m=1}^{\infty} a_m \sin(\mu_m x) = \sin\left(\frac{2\pi x}{L}\right),$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin(\mu_m x) = \sin\left(\frac{5\pi x}{L}\right).$$

Since we are hinted that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis, we now know that except for the following:

$$a_2 = 1 \text{ and } c\mu_5 b_5 = 1,$$

all the other coefficients are zero, so we have:

$$u(x,t) = \boxed{\cos\left(\frac{2\pi ct}{L}\right)\sin\left(\frac{2\pi x}{L}\right) + \frac{L}{5\pi c}\sin\left(\frac{5\pi ct}{L}\right)\sin\left(\frac{5\pi x}{L}\right)}.$$

9. Given a differential equation for $y = y(t)$ being:

$$t^3 y'' + ty' - y = 0.$$

(a) Verify that $y_1(t) = t$ is a solution to the differential equation.

(b)* Find the full set of solutions using reduction of order.

(c) Show that the set of solutions from part (b) is linearly independent.

Solution:

(a) *Proof.* We note that the left hand side is:

$$t^3 y_1'' + ty_1' - y_1 = t^3 \cdot 0 + t \cdot 1 - t = t - t = 0.$$

Hence $y_1(t) = t$ is a solution to the differential equation. □

(b) By reduction of order, we assume that the second solution is $y_2(t) = tu(t)$, then we plug $y_2(t)$ into the equation to get:

$$2t^3 u'(t) + t^4 u''(t) + tu(t) + t^2 u'(t) = t^4 u''(t) + (2t^3 + t^2)u'(t) = 0.$$

Here, we let $\omega(t) = u'(t)$ to get a first order differential equation:

$$t^2 \omega'(t) = (-2t - 1)\omega(t).$$

Clearly, this is separable, and we get that:

$$\frac{\omega'(t)}{\omega(t)} = -\frac{2t+1}{t^2} = -\frac{2}{t} - \frac{1}{t^2},$$

which by integration, we have obtained that:

$$\log\left(\omega(t)\right) = -2\log t + \frac{1}{t} + C.$$

By taking exponentials on both sides, we have:

$$\omega(t) = \exp\left(-2\log t + \frac{1}{t} + C\right) = \widetilde{C} e^{1/t} \cdot \frac{1}{t^2}.$$

Recall that we want $u(t)$ instead of $\omega(t)$, so we have:

$$u(t) = \int \omega(t) dt = \widetilde{C} \int e^{1/t} \cdot \frac{1}{t^2} dt = -\widetilde{C} e^{1/t} + D.$$

By multiplying t, we obtain that:

$$y_2 = -\widetilde{C} t e^{1/t} + Dt,$$

where Dt is repetitive in y_1, so we get:

$$y(t) = \boxed{C_1 t + C_2 t e^{1/t}}.$$

(c) *Proof.* We calculate Wronskian as:

$$\mathrm{W}[t, te^{1/t}] = \det \begin{pmatrix} t & te^{1/t} \\ 1 & e^{1/t} - \frac{e^{1/t}}{t} \end{pmatrix} = -e^{1/t} \neq 0,$$

hence the set of solutions is linearly independent. □

10.** Given the following second order initial value problem:

$$\begin{cases} \dfrac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \\ \dfrac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique.

Solution:
Note that I deliberately messed up with all the messy functions. Not only haven't I found a solution to the system, Wolfram cannot have an elementary solution as well. Hence, we need to think, alternatively, on some theorems.

Proof. Here, we suppose that $y(x)$ is a solution, and we want to show that $y(2-x)$ is also a solution. First we note that we can think of taking the derivatives of $y(2-x)$, by the chain rule:

$$\frac{d}{dx}[y(2-x)] = -y'(2-x),$$
$$\frac{d^2}{dx^2}[y(2-x)] = y''(2-x).$$

Now, if we plug in $y(2-x)$ into the system of equations, we have:

- First, for the differential equation, we have:

$$\begin{aligned} \frac{d^2}{dx^2}[y(x-2)] + \cos(1-x)y(x-2) &= y''(2-x) + \cos(x-1)y(2-x) \\ &= y''(2-x) + \cos\big(1-(2-x)\big)y(2-x) \\ &= y''(x) + \cos(1-x)y(x) \\ &= x^2 - 2x + 1 = (x-1)^2 = (1-x)^2 \\ &= \big((2-x)-1\big)^2 = (2-x)^2 - 2(2-x) + 1. \end{aligned}$$

- For the initial conditions, we trivially have that:

$$y(1) = y(2-1) \text{ and } y'(1) = y'(2-1).$$

Hence, we have shown that $y(2-x)$ is a solution if $y(x)$ is a solution.
Again, we observe the original initial value problem that:

$$\cos(1-x) \text{ and } x^2 - 2x + 1 \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that:

$$y(x) = y(2-x),$$

so the solution is symmetric about $x = 1$, as desired. □