Theorems and Formulas Booklet

Differential Equations

Summer 2024

Contents

Acknowledgments:

- The THEOREMS AND FORMULAS Booklets records the contents for `AS.110.302` Differential Equations and Applications at *Johns Hopkins University*.
- The notes summaries some contents, theorems, and formulas from the course as well as the textbook that is used:
  - *Elementary Differential Equations and Boundary Value Problems* by William E. Boyce, Richard C. Diprima, and Douglas B. Meade.
- The notes has been compiled and modified every term since Fall 2022 for PILOT by James Guo. It might contain minor typos or errors. Please point out any notable error(s).

Best regards, James Guo.
May 2024.

1 Preliminaries

1.1 Classifications of Differential Equations

Differential equations can be classified by their properties:

- Ordinary Differential Equations (ODEs) involves ordinary derivatives ($\frac{dy}{dt}$), while Partial Differential Equations (PDEs) involves partial derivatives ($\frac{\partial y}{\partial t}$).
- Single equation involves one unknown and one equation, while System of equations involves multiple unknowns and multiple equations.
- The order of the differential equation is the order of the highest derivatives term.
- Linear differential equations has only linear dependent on the function, while non-linear differential equations has non-linear dependent on the function.

1.2 Modeling Using ODEs

ODEs can be used for modeling. During modeling, it follows the following steps:

1. Construction of the Models,
2. Analysis of the Models,
3. Comparison of the Models with Reality.

1.3 Half Life Problems

The physics model for half life indicates the relationship between half life (τ) of a substance of amount $N(t)$ with initial amount N_0 at a time t is:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}},$$

where the rate of decay (λ) and half life (τ) are related by:

$$\tau \times \lambda = \log 2.$$

2 First Order ODEs

2.1 Integrating Factor

For ODEs in form $\frac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int a(t)dt\right).$$

2.2 Separable ODEs

For ODEs in form $M(t) + N(y)\frac{dy}{dt} = 0$, it can be separated by:

$$M(t)dt + N(y)dy = 0.$$

2.3 Existence and Uniqueness

The existence and uniqueness for Initial Value Problem (IVP) depend on cases:

- For an IVP in simple form:
$$\begin{cases} \dfrac{dy}{dt} = a(t) + b(t), \\ y(t_0) = y_0. \end{cases}$$
If $a(t)$ and $b(t)$ are continuous on an interval $[\alpha, \beta]$ and $t_0 \in [\alpha, \beta]$. Then, there exists a uniqueness solution y for $[\alpha, \beta]$ to the IVP.

- For an IVP in general form:
$$\begin{cases} \dfrac{dy}{dt} = f(t, y), \\ y(t_0) = y_0. \end{cases}$$
For $t_0 \in I = [a, b]$, $y_0 \in J = [c, d]$, if $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous on interval $I \times J = [a, b] \times [c, d]$. Then, there exists a unique solution on a smaller interval $I' \times J' \subset I \times J$.

2.4 Autonomous ODEs

Autonomous ODEs are in form of:
$$\frac{dy}{dt} = f(y).$$
The stability (stable/semi-stable/unstable) of equilibrium can be determined by phase lines, *i.e.*, the zeros of the function $f(t)$.

2.5 Logistic Population Growth

The logistic population growth model with population (y), growing rate (r), and carrying capacity (k) is given by:
$$\begin{cases} \dfrac{dy}{dt} = r\left(1 - \dfrac{y}{k}\right) y, \\ y(0) = y_0. \end{cases}$$
The solution for Logistic Population Growth is:
$$y(t) = \frac{ky_0}{(k - y_0)e^{-rt} + y_0}.$$

2.6 Exactness Problem

The condition for a function in form $M(x, y) + N(x, y)\dfrac{dy}{dx} = 0$ to be exact is:
$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$
For solving Exact ODEs, either finding $\int M(x, y)dx + h(y)$ or $\int N(x, y)dy + h(x)$ and taking partials again to fit gives the solution $\Psi(x, y) = C$.
For not exact cases, the integrating factor is:
$$\mu(t) = \exp\left(\int \frac{M_y - N_x}{N} dx\right) \qquad \text{or} \qquad \mu(t) = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

3 Second Order ODEs

3.1 Linear Homogeneous Cases

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is:

$$r^2 + pr + q = 0.$$

With solutions r_1 and r_2, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

If the solutions r_1 and r_2 are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 t e^{\lambda t} \sin(\beta t).$$

If the solutions r_1 and r_2 are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

3.2 Linear Independence

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

3.3 Existence and Uniqueness Theorem

Consider IVP in form:

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_1, y'(t_0) = y_2. \end{cases}$$

The interval I containing t_0 has $p(t)$, $q(t)$, and $g(t)$ continuous on it. Then, there is a unique solution $y(t)$ and twice differentiable on the interval I.

3.4 Superposition Theorem

If $y_1(t)$ and $y_2(t)$ are solutions to $l[y] = 0$, then the solution $c_1 y_1(t) + c_2 y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

3.5 Abel's Formula

Consider the equation $y'' + py' + qy = 0$, the Wronskian for the solutions are:

$$W[y_1, y_2] = C \exp\left(-\int p dt\right),$$

where C is independent of t but depends on y_1 and y_2.

3.6 Reduction of Order

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \times y_1(t).$$

3.7 Non-homogeneous Cases

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

- Undetermined Coefficient: A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Components in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig. Functions:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exponential Functions:	e^{at}	Ce^{at}

  Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: The particular solution is:

$$y_p = y_1(t) \int \frac{-y_2(t) \times g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \times g(t)}{W} dt.$$

4 Higher Order ODEs

4.1 Existence and Uniqueness Theorem

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t) y^{(n-1)} + \cdots + P_1(t) y' + P_0(t) y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots\ , y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t)$, $P_1(t)$, $\cdots$, $P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0. Then there exists a unique solution for $y(t)$ on I.

4.2 Homogeneous Cases

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

By computing the characteristic equation:

$$r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

With solutions r_1, r_2, $\cdots$, r_n, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}.$$

Note that the complex solutions can still be converted to sines and cosines, while repeated roots multiply a t on the repeated solutions.

4.3 Linear Independence

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}.$$

- Alternation to the Wronskian: By definition of linear independence, $f_1, f_2, \cdots, f_n$ are independent on I is equivalent to the expression where $k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$ if and only if $k_i = 0$.

4.4 Abel's Formula

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t) y^{(n-1)} + \cdots + P_1(t) y' + P_0(t) y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots, y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int P_{n-1}(t) dt},$$

where C is independent of t but depend on $y_1, y_2, \cdots, y_n$.

4.5 Non-Homogeneous Cases

Let the differential equation be:

$$L[y^{(n)}(t), y^{(n-1)}(t), \cdots, y(t)] = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$, $y_2(t)$, $\cdots$, $y_n(t)$ be all homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

- Undetermined Coefficient: Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Components in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig. Functions:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exponential Functions:	e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: The particular solution is:

$$y_p = y_1(t)\int \frac{W_1 g}{W}dt + y_2(t)\int \frac{W_2 g}{W}dt + \cdots + y_n(t)\int \frac{W_n g}{W}dt,$$

where W_i is defined to be the Wronskian with the i-th column alternated into $\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}$.

5 System of First Order Linear ODEs

5.1 Solving for Eigenvalues and Eigenvectors

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - Ir) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - Ir)\cdot \boldsymbol{\xi} = \mathbf{0}.$$

The solution to the ODE is:

$$\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t} + \cdots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}.$$

5.2 Linear Independence

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det\left(\Psi(t)\right).$$

The system is linearly independent if the Wronskian is non-zero.

5.3 Abel's Formula

For the linear system in form:

$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A, which is the sum of the diagonals, that is:

$$W = Ce^{\int \text{trace}\, A dt} = Ce^{\int (A_{1,1}+A_{2,2}+\cdots+A_{n,n})dt}.$$

5.4 Repeated Eigenvalues

For repeated eigenvalue r with only one eigenvector, if a given a solution is $\mathbf{x}^{(1)} = \boldsymbol{\xi}e^{rt}$, the other solution would be:

$$\mathbf{x}^{(2)} = \boldsymbol{\xi}te^{rt} + \boldsymbol{\eta}e^{rt},$$

where $(A - Ir)\cdot\boldsymbol{\eta} = \boldsymbol{\xi}$.

5.5 Fundamental Matrix

The exponential of Matrix is defined to be:

$$\exp(tA) = I + \sum_{n=1}^{\infty} \frac{(tA)^n}{n!},$$

where A^n is the result of n square matrices of A multiplying themselves.
The special case of fundamental matrix is defined to be Φ where:

$$\begin{cases} \Phi' = A \cdot \Phi, \\ \Phi(0) = I, \end{cases}$$

so that the fundamental matrix Φ can be calculated by:

$$\Phi(t) = \Psi(t) \cdot \Psi^{-1}(0).$$

5.6 Non-homogeneous Cases

Let the differential equation be:

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{g}(t),$$

where $\mathbf{g}(t)$ is a smooth vector-valued function. Let ϕ be its fundamental matrix, then the non-homogeneous cases can be solved by the following approaches:

- Diagonalization: Diagonalization utilizes T as the matrix of eigenvectors and D as the diagonal matrix of eigenvalues. Accordingly, let $\mathbf{x} = T\mathbf{y}$.
  Then, $\mathbf{x}' = T\mathbf{y}' = AT\mathbf{y} + \mathbf{g} = TD\mathbf{y} + \mathbf{g}$, which means that $\mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g}$ and the differential equation is now degenerated.

- Undetermined Coefficient: Same as in single equations, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $\mathbf{g}(t)$. Some brief strategies are:

Non-homogeneous Components in $\mathbf{g}(t)$	**Guess**
Polynomials: $\sum_{i=0}^{d} \mathbf{a_i} t^i$	$\sum_{i=0}^{d} \mathbf{c_i} t^i$
Trig. Functions: $\mathbf{a_1} \sin(b_1 t)$ and $\mathbf{a_2} \cos(b_2 t)$	$\mathbf{c_1} \sin(b_1 x) + \mathbf{c_2} \sin(b_2 x)$
Exponential Functions: $\mathbf{a} e^{bt}$	$\mathbf{c} e^{bt}$

  Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: Variation of parameters utilizes that:

$$\Psi \cdot \mathbf{u}' = \mathbf{g},$$

  where this equation can be solved by:

$$u_i' = \frac{W_i}{\det(\Psi)},$$

  where W_i is defined by the Wronskian of the matrix replacing the i-th column with $\mathbf{g}(t)$.
  There, the particular solution is:

$$\mathbf{x}_p = \Psi \cdot \mathbf{u}.$$

6 Non-linear Systems

6.1 Linear Approximation

For non-linear system $\mathbf{x}' = (\frac{F}{G})\mathbf{x}$, if $F, G \in C^2$, i.e. locally linear, the approximation at critical point (x_0, y_0) is:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \mathbf{J}(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where Jacobian is:

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}.$$

6.2 Autonomous Systems

When $\mathbf{x} = \begin{pmatrix} F(y) \\ G(x) \end{pmatrix}$, it can be solved implicitly for:

$$\frac{dy}{dx} = \frac{G(x)}{F(y)}.$$

6.3 Stability

For linearized system with 2 eigenvalues r_1, r_2, the following applies:

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$: $r_1 < r_2 < 0$ indicates an asymptotically stable node, $r_1 < 0 < r_2$ indicates a mostly unstable saddle, and $0 < r_1 < r_2$ indicates an unstable node. Note that these will not change for the non-linear case.

2. If $r_1 = r_2$: $r_1 = r_2 < 0$ indicates a asymptotically stable node and $r_1 = r_2 > 0$ indicates an unstable node. The stability preserves but the shape either node or spiral.

3. If $r_1, r_2 \in \mathbb{C}$ and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) \neq 0$: $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) > 0$ indicates an unstable spiral and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) < 0$ indicates an asymptotically stable spiral. Note that these will not change for the non-linear case.

4. If $r_1, r_2 \in \mathbb{C}$ and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) = 0$: That indicates a stable center. In the non-linear case, the shape is either spiral or center, but the stability is in-determinant.

6.4 Limit Cycles

A closed trajectory or periodic solution repeats back to itself with period τ:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Note that the closed trajectories with either side converging to the solution is a limit cycle.

6.5 Conversion to Polar Coordinate

A Cartesian coordinate can be converted by:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ rr' = xx' + yy', \\ r^2\theta' = xy' - yx'. \end{cases}$$

For a linear system $x = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$ with $F, G \in C^1$:

1. A closed trajectory of the system must enclose at least 1 critical point.
2. If it only encloses 1 critical point, then that critical point cannot be saddle point.
3. If there are no critical points, there are no closed trajectories.
4. If the unique critical point is saddle, there are no trajectories.
5. For a simple connected domain D in the xy-plane with no holes. If $F_x + G_y$ had the same sign throughout D, then there is no closed trajectories in D.

7 Laplace Transformation

7.1 Properties of Laplace Transformation:

The Laplace Transformation of a function f is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Note that Laplace Transformation can be used on non-continuous functions by utilizing step functions. Laplace Transformation has the following properties:

1. Laplace Transformation is a linear operator:
$$\mathcal{L}\{f + \lambda g\} = \mathcal{L}\{f\} + \lambda\mathcal{L}\{g\}$$
2. Laplace Transformation for derivatives:
$$\begin{aligned} \mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0), \\ \mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0), \\ &\vdots \\ \mathcal{L}\{f^{(n)}(t)\} &= s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0). \end{aligned}$$
3. First Shifting Theorem:
$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c).$$

The Laplace Transformations can be used for solving IVP, where the inverse helps to find the original function prior to transformation.

7.2 Elementary Laplace Transformations

The Laplace Transformations for elementary functions are given in the following table, note that they can still be calculated by its definition:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$t^n, n \in \mathbb{Z}_{>0}$	$\frac{n!}{s^{n+1}}, s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}, s > 0$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

7.3 Step Functions:

The step functions are defined by:

$$u_c(t) = u(t-c) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

And the Laplace Transformations of the step function is:

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}.$$

The step function forms the Second Shifting Theorem:

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$$

7.4 Impulse Functions

The idealized unit impulse function $\delta(t)$, or *Dirac delta function*, satisfies the properties that:

$$\delta(t) = 0 \text{ for } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

There is no ordinary function satisfying the idealized unit impulse function, so it is a generalized function. A unit impulse at an arbitrary point $t = t_0$, denoted by $\delta(t - t_0)$, follows that:

$$\delta(t) = 0 \text{ for } t \neq t_0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t - t_0) dt = 1.$$

The Laplace Transformation of the impulse function is:

$$\mathcal{L}\{\delta(t - c)\} = e^{-cs}.$$

7.5 Convolution

The convolution of f and g, denoted $(f * g)$, is defined as:

$$(f * g)(t) = \int_0^t f(t - \tau) g(\tau) d\tau = \int_0^t f(\tau) g(t - \tau) d\tau.$$

The convolution $f * g$ has many of the properties of ordinary multiplication:

1. Commutativity: $f * g = g * f$;
2. Distributivity: $f * (g + h) = f * g + f * h$;
3. Associativity: $(f * g) * h = f * (g * h)$;
4. Zero Property: $f * 0 = 0 * f = 0$, where 0 is a function that maps any input to 0.

The Laplace Transformation of the convolution of f and g is:

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s).$$

8 Numerical Methods

8.1 Euler's Method

The numerical approximation focuses on first-order initial value problem:

$$\begin{cases} \dfrac{dy}{dt} = f(t, y), \\ y(t_0) = y_0. \end{cases}$$

By the *Existence and Uniqueness Theorem*, a unique solution exists for some rectangular region containing (t_0, y_0) when f and $\frac{\partial f}{\partial y}$ are continuous. With this foundation, we may apply Euler's method on such region. (*Note that* out of the region, the approximation would not be accurate.)
Euler's method recursively applies the following function:

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n), \;\; n = 0, 1, 2, \cdots,$$

and when the steps are constrained to be a constant h, we have:

$$y_{n+1} = y_n + hf(t_n, y_n), \;\; n = 0, 1, 2, \cdots.$$

Typically, Euler's method incurs error, whereas some typical issues are:

1. When the step size h is too big, the error is significant.
2. When the step size h is too small, the cost of calculation is expensive.
3. The computation does address the asymptotic behaviors.
4. When the vector field has steep components, the approximation differs more.

8.2 Generalization on Euler's Method

Euler's method can be analyzed by using the *Fundamental Theorem of Calculus*, that is:

$$\begin{aligned} y(t) &= y(t_n) + \int_{t_n}^{t} f\big(s, y(s)\big) ds \\ &\approx y(t_n) + \sum_{t_0 \le t_i < t_{i+1} \le t} f(t_i, y_n)(t_{i+1} - t_i), \end{aligned}$$

in which we may establish the improved Euler's Method, by:

$$y_{n+1} = y_n + h\left(\frac{f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))}{2}\right),$$

by considering the trapezoid approach for *Riemann sum*.
Since the $f(t, y)$ depends only on t and not on y, then solving differential equation reduced from $y' = f(t, y)$ to integrating $f(t)$, which makes the improved Euler's Method into:

$$y_{n+1} = y_n + \frac{h}{2}\big(f(t_n) + f(t_n + h)\big).$$