PILOT

Exam 1 Review Problem Set 1: Solutions

Differential Equations

Summer 2025

1. Solve the following initial value problem (IVP) on $y = y(x)$, and specify the domain for your solution:

$$\begin{cases} y' = (x \log x)^{-1}, \\ y(e) = -6. \end{cases}$$

Solution:

Here, we notice that this problem is separable, hence we can write:

$$dy = \frac{1}{x \log x} dx,$$

$$\int dy = \int \frac{1}{x \log x} dx.$$

Now, we evaluate the integral by substitution, *i.e.*, $u = \log x$ and $du = dx/x$, which give that:

$$y = \int \frac{1}{u} du = \log |u| + C = \log |\log x| + C.$$

Eventually, we plug in the initial condition, that is $y(e) = -6$, giving us that:

$$-6 = \log |\log e| + C,$$

$$C = -6.$$

Therefore, the solution is:

$$y = \boxed{\log |\log x| - 6}.$$

Here, we note that $\log(-)$ has a valid domain over positive numbers, and the double $\log(-)$ functions enforces that x must be greater than 1, as $\log(0)$ is undefined. Since our initial condition is e, and $e \in (1, \infty)$, the domain of the solution is $\boxed{(1, \infty)}$.

2. Suppose $f(x)$ is non-zero, let an initial value problem be:

$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

(a) Show that the differential equation is **not** linear.

For the next two questions, suppose $f(x) = \tan x$.

(b) State, without justification, the open interval(s) in which $f(x)$ is continuous.

(c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

(d) Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

Solution:

(a) *Proof.* We can write the equation as $F(x, y, y') := y' - \dfrac{xf(x)}{(y+1)(y-1)} = 0$, and since:

$$F\big(x, (y+1), (y+1)'\big) = y' - \frac{xf(x)}{(y+2)y} \neq 1,$$

so the function is non-linear. □

(b) Here, we should consider that:

$$f(x) = \tan x = \frac{\sin x}{\cos x},$$

so the discontinuities are at when $\cos x = 0$, that is:

$$x \in \left\{ \frac{(2k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Hence, we have the intervals in which $f(x)$ being continuous as:

$$\boxed{\left\{ \left(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2} \right) : k \in \mathbb{Z} \right\}}.$$

(c) *Proof.* Here, we want to write our equation in the standard form and obtain that:

$$y' := f(t, y) = \frac{x \tan x}{(y+1)(y-1)}, \qquad \frac{\partial f(t,y)}{\partial y} \qquad = -\frac{x \tan x \cdot 2y}{(y^2-1)^2}.$$

Clear, we note the discontinuities of y at $y = \pm 1$, and x demonstrated as above, thus we can form a rectangle $Q = (-\pi/2, \pi/2) \times (-1, 1)$ in which the initial condition $(0, 0) \in Q$ and $f(t, y)$ with $\partial_y f(t, y)$ are continuous on the interval. By the *existence and uniqueness theorem for non-linear case*, we know that there exists some δ such that there is a unique solution for $-\delta < x < \delta$. □

(d) If the condition in (c) does not hold, by contraposition, this implies that continuity must fail, *i.e.*, $\boxed{xf(x) \text{ must be discontinuous at } x = 0}$. Hence, some examples could be:

$$f(x) = \frac{1}{x^2}, \text{ or } \log x, \text{ or } \csc x, \text{ or } \chi_{\{0\}}(x) \text{ etc.}$$

3. Draw the phase line and determine the stability of each equilibrium for the following autonomous differential equations:

(a)
$$y' = y^4 - 3y^3 + 2y^2.$$

(b)
$$y' = y^{2025} - 1.$$

Solution:

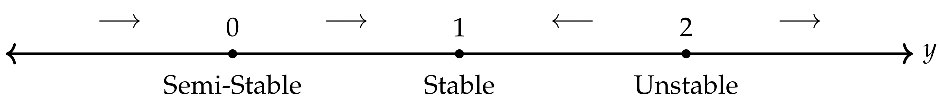
(a) First, we need to factor the right hand side polynomial as:

$$y^4 - 3y^3 + 2y^2 = y^2(y^2 - 3y + 2) = y^2(y-1)(y-2),$$

so, we can trivially note the roots as:

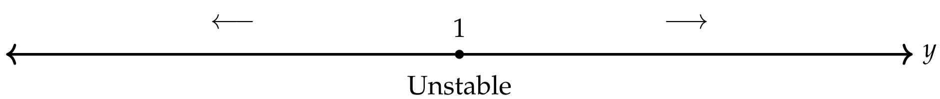
$$y = 0 \text{ with multiplicity } 2, y = 1, \text{ and } y = 2.$$

Sophisticated readers shall notice that this polynomial has a positive leading coefficient, hence it approaches $+\infty$ when $y \to \infty$, hence the arrows can be easily determined.
Otherwise, readers can plug in a value within each intervals, such as $y = 3$ for $y > 2$, $y = 3/2$ for $1 < y < 2$, etc., which should work equivalently.
Hence, we should expect a graph as follows:



The stability is given by the directions of the arrows.

(b) Here, readers shall realize that the right hand side polynomial is monotonic ($(y^{2025} - 1)' \geq 0$), so the only real root is at $y = 1$, and since the polynomial has positive leading coefficient and odd order, it shall approach $\mp\infty$ as $y \to \mp\infty$, so the phase line is:



The stability is given by the directions of the arrows.

4. Let a differential equation be defined as follows:

$$\frac{dy}{dx} = e^{2x} + y - 1.$$

(a) What is the integrating factor ($\mu(x)$) for the equation? Solve for the general solution.

(b) Is the equation *exact*? If not, make it exact, then find the general solution.

(c) Do solutions from part (a) and (b) agree?

Solutions:

(a) First, we write the equation in standard form, that is $y' - y = e^{2x} - 1$. Hence, with $p(x) = -1$, the integrating factor is:

$$\mu(x) = \exp\left(\int_0^x p(s)ds\right) = \exp\left(\int_0^x (-1)ds\right) = \boxed{\exp(-x)}.$$

Then, we multiply the integrating factors on both ends to obtain:

$$\begin{aligned} \frac{d}{dx}\left[ye^{-x}\right] &= e^x - e^{-x}, \\ ye^{-x} &= \int \left(e^x - e^{-x}\right) dx = e^x + e^{-x} + C, \\ y &= \boxed{Ce^x + e^{2x} + 1}. \end{aligned}$$

(b) Note that for exactness, we write the equation as:

$$\underbrace{(-e^{2x} - y + 1)}_{M(x,y)} + \underbrace{(1)}_{N(x,y)} \frac{dy}{dx} = 0,$$

meaning that their partial derivatives are, respectively:

$$\partial_y M(x,y) = -1 \text{ and } \partial_x N(x,y) = 0,$$

and since they are different, the equation is $\boxed{\text{not exact}}$.

Thus, we look for the integrating factor, *i.e.*:

$$\mu(t) = \exp\left(\int_0^x \frac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}\right) = \exp\left(\int_0^x \frac{-1-0}{1} ds\right) = \exp(-x).$$

Now, we multiply e^{-x} on both sides, giving us that:

$$\underbrace{(-e^x - ye^{-x} + e^{-x})}_{\tilde{M}(x,y)} + \underbrace{(e^{-x})}_{\tilde{N}(x,y)} \frac{dy}{dx} = 0.$$

Now, the equation should be exact. *We leave the check to the readers as an exercise.*

To get the solution, we first integrate $\tilde{M}(x,y)$ with respect to x, that is:

$$\varphi(x,y) = \int (-e^x - ye^{-x} + e^{-x})dx = -e^x + ye^{-x} - e^{-x} + h(y).$$

Now, taking the derivative with respect to y gives:

$$\partial_y \varphi(x,y) = e^{-x} + h'(y) = e^{-x},$$

which pushes $h(y)$ to be constant, hence we have solution:

$$\varphi(x,y) = \boxed{-e^x + ye^{-x} - e^{-x} = C}.$$

(c) The solutions $\boxed{\text{agree}}$ by simple arithmetic deductions.

5.* This brief digression to "differential forms" aims for the following goals:

- Legitimize $\dfrac{\partial y}{\partial x} = \dfrac{f(x)}{g(y)} \iff g(y)dy = f(x)dx$ via the differential operator d.
- Get the foundation of *exactness* for certain differential equation relationship.

First, consider variables $x_1, x_2, \cdots, x_n$, we may defined the wedge product ($\wedge$) to connect any two variables satisfying that:

$$x_i \wedge x_j = -x_j \wedge x_i \text{ for all } 1 \le i, j \le n.$$

(a) Show that $x_i \wedge x_i = 0$ for $1 \le i \le n$.

Now, given any smooth function f, we defined the differential operator (d) as:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

(b) Suppose $y(x) = e^x$, find dy.

(c) Now, suppose that $\frac{\partial y}{\partial x} = \frac{f(x)}{g(y)}$, can you express dy in terms of the differential form of x.
Note: Since we have just one variable, we have $dy/dx = \partial y/\partial x$, leading to our first goal.

Furthermore, we can apply the differential operator over differential forms with wedge products already. Suppose:

$$\omega = \sum_{i_1, \cdots, i_k} f_{i_1, \cdots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

we may have the differential of ω as:

$$d\omega = \sum_{i_1, \cdots, i_k} (df_{i_1, \cdots, i_k}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

(d) Suppose x, y are the variables, and $\omega = 2xy^2 dx + 2x^2 y dy$, show that $d\omega = 0$.

This then relates to a concept called *exactness* in differential equations. Consider the equation:

$$\frac{dy}{dx} + \frac{F(x,y)}{G(x,y)} = 0,$$

we can rewrite it as $F(x,y)dx + G(x,y)dy = 0$. Exactness enforces that:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}.$$

Similarly, exactness is considering finding a solution $f(x,y) = c$ such that $F = \frac{\partial f}{\partial x}$ and $G = \frac{\partial f}{\partial y}$.

(e) Show that $df = F(x,y)dx + G(x,y)dy$ and exactness is equivalently $d(df) = 0$.
Note: This implies that the differential equation in part (d) satisfies *exactness*.

Solution:

(a) *Proof.* Since we have:

$$x_i \wedge x_i = -x_i \wedge x_i,$$

we must have $x_i \wedge x_i = 0$. □

(b) By the given differential operator:

$$dy = \frac{\partial y}{\partial x} dx = \boxed{e^x} dx.$$

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(c) Then, we have:

$$dy = \frac{\partial y}{\partial x}dx = \boxed{\frac{f(x)}{g(y)}dx}.$$

Hence, we justify the separation of the variables as $g(y)dy = f(x)dx$.

(d) *Proof.* As instructed, we have:

$$\begin{aligned} d\omega &= \frac{\partial}{\partial x}(2xy^2)dx \wedge dx + \frac{\partial}{\partial y}(2xy^2)dy \wedge dx + \frac{\partial}{\partial x}(2x^2y)dx \wedge dy + \frac{\partial}{\partial y}(2x^2y)dy \wedge dy \\ &= 0 + 4xydy \wedge dx + 4xydx \wedge dy + 0 = -4xydx \wedge dy + 4xydx \wedge dy = 0, \end{aligned}$$

as desired. □

(e) *Proof.* First, we have that:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Fdx + Gdy.$$

Then, in terms of the exactness relationship, we have:

$$\begin{aligned} \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} &\iff \frac{\partial F}{\partial y}dy \wedge dx = -\frac{\partial G}{\partial x}dx \wedge dy \\ &\iff \frac{\partial F}{\partial y}dy \wedge dx + \frac{\partial G}{\partial x}dx \wedge dy = 0 \\ &\iff \frac{\partial F}{\partial x}dx \wedge dx + \frac{\partial F}{\partial y}dy \wedge dx + \frac{\partial G}{\partial x}dx \wedge dy + \frac{\partial G}{\partial y}dy \wedge dy = 0 \\ &\iff d(df) = 0. \end{aligned}$$

Hence, we have shown that the exactness is exactly that the differential form satisfies that $d(df) = 0$. □

In fact, for any smooth function f, we have $d(df) = 0$, which is the equivalent of the conclusion such that mixed partials are equal. We invite capable readers to investigate that $d^2 := d \circ d = 0$ for all smooth function f. Additionally, people with experiences in vector calculus could investigate the following *commutative diagram*.

$$\begin{array}{ccccccc} \Omega^0(\mathbb{R}^3) & \xrightarrow{d(-)} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d(-)} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d(-)} & \Omega^3(\mathbb{R}^3) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ C^\infty(\mathbb{R}^3, \mathbb{R}) & \xrightarrow{\nabla(-)} & C^\infty(\mathbb{R}^3, \mathbb{R}^3) & \xrightarrow{\nabla\times(-)} & C^\infty(\mathbb{R}^3, \mathbb{R}^3) & \xrightarrow{\nabla\cdot(-)} & C^\infty(\mathbb{R}^3, \mathbb{R}) \end{array}$$

The above are respectively 0-form, 1-form, 2-form, and 3-form (with 0, 1, 2, or, 3 $\wedge$'s in the differential form) and the below are smooth functions mapping in respective Euclidean spaces.

6.* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:,

$$y'(x) = e^{2x} + y(x) - 1.$$

Solution:
First, we write the equation in the general form:

$$\frac{dy}{dx} + (1 - e^{2x} - y) = 0.$$

Now, we take the partial derivatives to obtain that:

$$\frac{\partial}{\partial y}[1 - e^{2x} - y] = -1,$$

$$\frac{\partial}{\partial x}[1] = 0.$$

Notice that the mixed partials are not the same, the equation is $\boxed{\text{not exact}}$.
Here, we choose the integrating factor as:

$$\begin{aligned}\mu(x) &= \exp\left(\int_0^x \frac{\frac{\partial}{\partial y}[1 - e^{2s} - y] - \frac{\partial}{\partial s}[1]}{1} ds\right)\\ &= \exp\left(\int_0^x -ds\right) = \exp(-x).\end{aligned}$$

Therefore, our equation becomes:

$$(e^{-x})\frac{dy}{dx} + (e^{-x} - e^{x} - ye^{-x}) = 0.$$

After multiplying the integrating factor, it would be exact. *We leave the repetitive check as an exercise to the readers*.
Now, we can integrate to find the solution with a $h(y)$ as function:

$$\varphi(x,y) = \int (e^{-x} - e^{x} - ye^{-x})dx = -e^{-x} - e^{x} + ye^{-x} + h(y).$$

By taking the partial derivative with respect to y, we have:

$$\partial_y \varphi(x,y) = e^{-x} + h'(y),$$

which leads to the conclusion that $h'(y) = 0$ so $h(y) = C$.
Then, we can conclude that the solution is now:

$$\varphi(x,y) = -e^{-x} - e^{x} + ye^{-x} + C = 0,$$

which is equivalently:

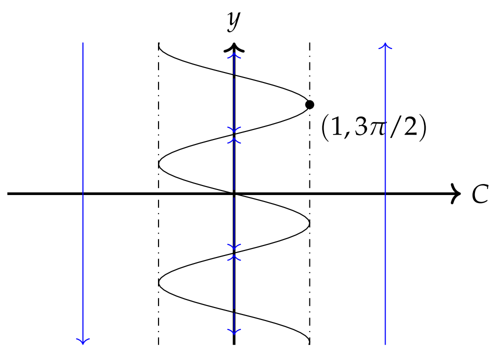
$$y(x) = \boxed{\widetilde{C}e^{x} + 1 + e^{2x}}.$$

7. For the first-order autonomous ODE:

$$\frac{dy}{dt} = \sin y + C,$$

where $C \in \mathbb{R}$ is a parameter. Determine any and all bifurcation values for the parameter C and sketch a bifurcation diagram.

Solution:
It is not hard to observe that $\sin y$ will intersect the axis infinitely many times, while $\sin(\mathbb{R}) = [-1, 1]$, one shall then realize that the bifurcation value would be ± 1, since when $C > 1$ or $C < -1$, there will be no equilibriums at all.
Therefore, the bifurcation diagram can be illustrated as:



8. Let a first order IVP on $y := y(t)$ be defined as follows:

$$\begin{cases} y' = \frac{2}{t}y, \\ y(1) = 1. \end{cases}$$

(a) Find the solution to the above initial value problem.

(b) Recall the theorem on existence and uniqueness, as follows:

> For an IVP in simple form:
>
> $$\begin{cases} \frac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$
>
> For some $I = (\alpha, \beta) \ni t_0$, if $a(t)$ and $b(t)$ are continuous on the interval I. Then, there exists a unique solution to the IVP on the interval I.

Show that the IVP in this problem does not satisfy the condition for the existence and uniqueness theorem for $\mathbb{R}$.

(c) Does the above example violates the existence and uniqueness theorem? Why?

Solution:

(a) This problem is clearly separable, we may compute:

$$\frac{dy}{y} = 2\frac{dt}{t}$$
$$\int \frac{dy}{y} = 2\int \frac{dt}{t}$$
$$\log|y| = 2\log|t| + C$$
$$y = \tilde{C}t^2.$$

Note that the initial condition enforces that $y(1) = 1$, so the solution is just:

$$y = \boxed{t^2}.$$

(b) Note that $a(t) = 2/t$, which is not continuous over $(-\infty, 0) \cup (0, \infty)$, then the theorem does not guarantee the existence and uniqueness of a solution over $\mathbb{R}$.

(c) This is not a violation since the converse of the theorem is not necessarily true. In propositional logic, if A implies B (written as $A \implies B$), the converse (B implies A, written as $B \implies A$) is not necessarily true. Hence, we can still have a solution that is unique over $\mathbb{R}$.

9. Solve the following second order differential equations for $y = y(x)$:

(a) $$y'' + y' - 132y = 0.$$

(b) $$y'' - 4y' = -4y.$$

(c) $$y'' - 2y' + 3y = 0.$$

Solution:

(a) We find the characteristic polynomial as $r^2 + r - 132 = 0$, which can be trivially factorized into:

$$(r - 11)(r + 12) = 0,$$

so with roots $r_1 = 11$ and $r_2 = -12$, we have the general solution as:

$$y(x) = \boxed{C_1 e^{11x} + C_2 e^{-12x}}.$$

(b) We turn the equation to the standard form:

$$y'' - 4y' + 4 = 0.$$

We find the characteristic polynomial as $r^2 - 4r + 4 = 0$, which can be immediately factorized into:

$$(r - 2)^2 = 0,$$

so with roots $r_1 = r_2 = 2$ (repeated roots), we have the general solution as:

$$y(x) = \boxed{C_1 e^{2x} + C_2 x e^{2x}}.$$

(c) We find the characteristic polynomial as $r^2 - 2r + 3 = 0$, which the quadratic formula gives:

$$r = \frac{2 \pm \sqrt{2^2 - 4 \times 3}}{2} = 1 \pm \mathrm{i}\sqrt{2}$$

so with roots $r_1 = 1 + \mathrm{i}\sqrt{2}$ and $r_2 = 1 - \mathrm{i}\sqrt{2}$, we would have the solution:

$$y(x) = C_1 e^{(1+\mathrm{i}\sqrt{2})x} + C_2 e^{(1-\mathrm{i}\sqrt{2})x}.$$

To obtain real solution, we apply Euler's identity:

$$y_1(x) = e^x\big(\cos(\sqrt{2}x) - \mathrm{i}\sin(\sqrt{2}x)\big) \text{ and } y_2(x) = e^x\big(\cos(-\sqrt{2}x) - \mathrm{i}\sin(-\sqrt{2}x)\big).$$

By the *principle of superposition*, we can linearly combine the solutions to be different solutions, so we have:

$$\widetilde{y_1}(x) = \frac{1}{2}(y_1 + y_2) = e^x \cos(\sqrt{2}x),$$
$$\widetilde{y_2}(x) = \frac{1}{2}(y_2 - y_1) = e^x \sin(\sqrt{2}x).$$

One can verify that $\widetilde{y_1}$ and $\widetilde{y_2}$ are linearly independent by taking Wronskian,*i.e.*:

$$\begin{aligned} \mathrm{W}[\widetilde{y_1}, \widetilde{y_2}] &= \det\begin{pmatrix} e^x \cos(\sqrt{2}x) & e^x \sin(\sqrt{2}x) \\ e^x \cos(\sqrt{2}x) - \sqrt{2}e^x \sin(\sqrt{2}x) & e^x \sin(\sqrt{2}x) + \sqrt{2}e^x \cos(\sqrt{2}x) \end{pmatrix} \\ &= \sqrt{2}e^{2x}\cos^2(\sqrt{2}x) + \sqrt{2}e^{2x}\sin^2(\sqrt{2}x) = \sqrt{2}e^{2x} \neq 0. \end{aligned}$$

Now, they are linearly independent, so we have the general solution as:

$$y(x) = \boxed{C_1 e^x \cos(\sqrt{2}x) + C_2 e^x \sin(\sqrt{2}x)}.$$

10.* Given the following second order initial value problem:

$$\begin{cases} \frac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \\ \frac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.

Hint: Consider the interval in which the solution is unique.

Solution:

Note that I deliberately messed up with all the messy functions. Not only haven't I found a solution to the system, Wolfram cannot have an elementary solution as well. Hence, we need to think, alternatively, on some theorems.

Proof. Here, we suppose that $y(x)$ is a solution, and we want to show that $y(2-x)$ is also a solution. First we note that we can think of taking the derivatives of $y(2-x)$, by the chain rule:

$$\frac{d}{dx}[y(2-x)] = -y'(2-x),$$
$$\frac{d^2}{dx^2}[y(2-x)] = y''(2-x).$$

Now, if we plug in $y(2-x)$ into the system of equations, we have:

- First, for the differential equation, we have:

$$\begin{aligned} \frac{d^2}{dx^2}[y(x-2)] + \cos(1-x)y(x-2) &= y''(2-x) + \cos(x-1)y(2-x) \\ &= y''(2-x) + \cos\big(1-(2-x)\big)y(2-x) \\ &= y''(x) + \cos(1-x)y(x) \\ &= x^2 - 2x + 1 = (x-1)^2 = (1-x)^2 \\ &= \big((2-x)-1\big)^2 = (2-x)^2 - 2(2-x) + 1. \end{aligned}$$

- For the initial conditions, we trivially have that:

$$y(1) = y(2-1) \text{ and } y'(1) = y'(2-1).$$

Hence, we have shown that $y(2-x)$ is a solution if $y(x)$ is a solution.

Again, we observe the original initial value problem that:

$$\cos(1-x) \text{ and } x^2 - 2x + 1 \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that:

$$y(x) = y(2-x),$$

so the solution is symmetric about $x = 1$, as desired. □