Exam 1 Review Problem Set 2: Solutions

Differential Equations

Summer 2025

1. Solve for the general solution to the following ODEs with $y = y(t)$:

   (a) $$2y' + y = 3t.$$

   (b) $$y' + \log(t)y = t^{-t}.$$

Solution:

(a) Here, we first convert the equation to standard form, *i.e.*:

$$y' + \frac{1}{2}y = \frac{3}{2}t.$$

Hence, with $p(t) = 1/2$, the integration factor must be:

$$\mu(t) = \exp\left(\int_0^t p(s)ds\right) = \exp\left(\int_0^t \frac{1}{2}ds\right) = \exp\left(\frac{1}{2}t\right).$$

Now, we multiply the integration factor on both sides, giving that:

$$\begin{aligned}
y'e^{t/2} + \frac{1}{2}ye^{t/2} &= \frac{3}{2}te^{t/2}, \\
\frac{d}{dt}\left[e^{t/2}y\right] &= \frac{3}{2}te^{t/2}, \\
e^{t/2}y &= \frac{3}{2}\int te^{t/2}dt = \frac{3}{2}\left[2te^{t/2} - \int 2e^{t/2}\right] \\
&= \frac{3}{2}\left[2te^{t/2} - 4e^{t/2} + C\right] = 3te^{t/2} - 6e^{t/2} + \tilde{C}, \\
y &= \boxed{\tilde{C}e^{-t/2} + 3t - 6}.
\end{aligned}$$

(b) Again, we find the integration factor as:

$$\mu(t) = \exp\left(\int_e^t \log s ds\right).$$

To find the antiderivative of $\log(-)$, we have:

$$\int \log s ds = \int 1 \cdot \log s ds = s\log s - \int s \cdot \frac{1}{s}ds = s\log s - \int ds = s\log s - s + C.$$

Therefore, the integrating factor is:

$$\mu(t) = \exp(t\log t - t) = \frac{\exp(t\log t)}{e^t} = \frac{t^t}{e^t}.$$

Hence, we multiply the integration factor on both sides, giving that:

$$\begin{aligned}
\frac{t^t}{e^t} + \frac{t^t}{e^t}\log ty &= e^{-t}, \\
\frac{d}{dt}\left[\frac{t^t}{e^t}y\right] &= e^{-t}, \\
\frac{t^t}{e^t}y &= \int e^{-t}dt = -e^{-t} + C, \\
y &= \boxed{-t^{-t} + Ct^{-t}e^t}.
\end{aligned}$$

2. Solve the following initial value problem (IVP) on $y = y(x)$, and specify the domain for your solution:
$$\begin{cases} y' = y(y+1), \\ y(0) = 1. \end{cases}$$

Solution: Now, we trivially separate the problem as:
$$\frac{dy}{y(y+1)} = dx.$$
Then, we use the partial fraction to obtain that:
$$\left(\frac{1}{y} - \frac{1}{y+1}\right) dy = dx.$$
Now, we can integrate both side to obtain that:
$$\begin{aligned} \int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy &= \int dx, \\ \log|y| - \log|y+1| &= x + C, \\ \log\left|\frac{y}{y+1}\right| &= x + C, \\ \frac{y}{y+1} &= \tilde{C}e^x. \end{aligned}$$
Now, we consider the left hand side as:
$$\frac{y}{y+1} = \frac{y+1}{y+1} - \frac{1}{y+1} = 1 - \frac{1}{y+1},$$
which allows us to rewrite the equation as:
$$\begin{aligned} \frac{1}{y+1} &= 1 - \tilde{C}e^x, \\ y &= \frac{1}{1-\tilde{C}e^x} - 1 = \frac{\tilde{C}e^x}{1-\tilde{C}e^x}. \end{aligned}$$
By plugging in the initial condition, we trivially have:
$$1 = \frac{\tilde{C}}{1-\tilde{C}} \implies \tilde{C} = \frac{1}{2}.$$
Hence, the solution is:
$$y = \boxed{\frac{e^x}{2-e^x}}.$$
For the domain of our solution, we note that e^x is continuous, but $2 - e^x$ could cause a zero denominator at $x = \log 2$. Note that $\log 2 > 0$, so the domain of the solution is $\boxed{(-\infty, \log 2)}$.

3. Given an initial value problem:

$$\begin{cases} \dfrac{dy}{dt} - \dfrac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$

(a) Find the integrating factor $\mu(t)$.

(b) Solve for the particular solution for the initial value problem.

(c) Discuss the behavior of the solution as $t \to \infty$ for different cases of y_0.

Solution:

(a) As instructed, we look for the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t -\frac{3}{2}ds\right) = \boxed{\exp\left(-\frac{3}{2}t\right)}.$$

(b) With the integrating factor, we multiply both sides by $\mu(t)$ to obtain that:

$$y'e^{-3t/2} - \frac{3}{2}ye^{-3t/2} = 3te^{-3t/2} + 2e^t e^{-3t/2}.$$

Clearly, we observe that the left hand side is the derivative after product rule for $ye^{-3t/2}$ and the right hand side can be simplified as:

$$\frac{d}{dt}\left[ye^{-3t/2}\right] = 3te^{-3t/2} + 2e^{-t/2}.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{-3t/2} &= \int 3te^{-3t/2}dt + \int 2e^{-t/2}dt \\ &= -2te^{-3t/2} + 2\int e^{-3t/2}dt - 4r^{-t/2} + C \\ &= -2te^{-3t/2} - \frac{4}{3}e^{-3t/2} - 4r^{-t/2} + C. \end{aligned}$$

Then, we divide both sides by $e^{-3t/2}$ to get the general solution:

$$y(t) = -2t - \frac{4}{3} - 4e^t + Ce^{3t/2}.$$

Given the initial condition, we have that:

$$y_0 = 0 - \frac{4}{3} - 4 + C,$$

which implies $C = 16/3 + y_0$, leading to the particular solution that:

$$y(t) = \boxed{-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}}.$$

(c) We observe that:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}\right].$$

Note that the important terms are e^t and $e^{3t/2}$, we need to care the critical value $-16/3$:

- when $y_0 > -16/3$, $\boxed{y(t) \to \infty \text{ when } t \to \infty}$,
- when $y_0 \leq -16/3$, $\boxed{y(t) \to -\infty \text{ when } t \to \infty}$.

4. An autonomous differential equation is given as follows:

$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \text{ where } t \geq 0 \text{ and } y \geq 0.$$

Draw a phase portrait and sketch a few solutions with different initial conditions.

Solution:

Recall from Pre-Calculus (or Algebra) the following *Rational root test*:

Theorem: Rational Root Test. Let the polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$

have integer coefficients $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$, then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p|a_0$ and $q|a_n$. ⌟

From the theorem, we can note that if the equation has a rational root, it must be one of:

$$r = \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}.$$

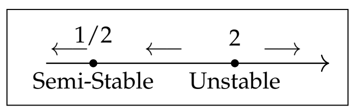
By plugging in, one should notice that $y = 2$ is a root (one might also notice $1/2$ is a root as well, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:

$$\frac{4y^3 - 12y^2 + 9y - 2}{y - 2} = 4y^2 - 4y + 1.$$

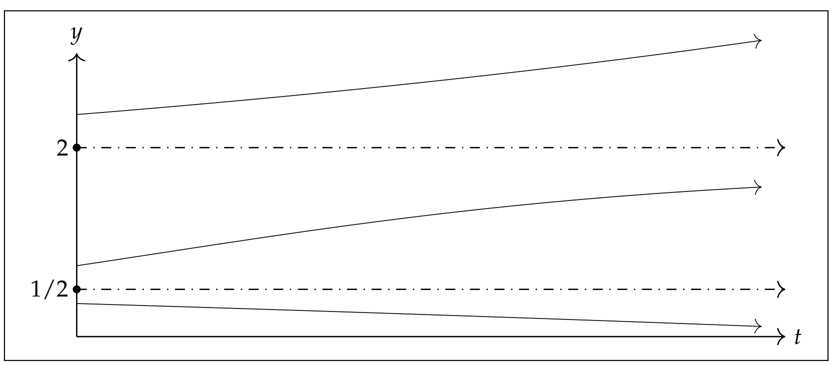
Clear, we can notice that the right hand side is a perfect square (else, you could use the quadratic formula) that:

$$4y^2 - 4y + 1 = (2y - 1)^2.$$

Thus, we now know that the roots are 2 and $1/2$ (multiplicity 2). Hence, the phase portrait is:



Correspondingly, we can sketch a few solutions (not necessarily in scale):



Note that the **Theorem** can also be generalized into the following manner (in ring theory):

Theorem: Rational Root Theorem. Let R be UFD, and polynomial:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in R[x],$$

and let $r = p/q \in K(R)$ be a root of f with $p, q \in R$ and $\gcd(p, q) = 1$, then $p|a_0$ and $q|a_n$. ⌟

The proofs of the **Theorems** left as exercises to diligent readers.

5. Let a differential equation be defined as:
$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$
Use Euler's Method with step size $h = 1$ to approximate $y(5)$.

Solution:
With $y(0) = 0$, we have $y'(0) = 0 - 0 = 0$. We do the following steps:

- We approximate $y(1) \approx y(0) + 1 \cdot y'(0) = 0$, then we have $y'(1) \approx 1 - 0 = 1$.
- We approximate $y(2) \approx y(1) + 1 \cdot y'(1) \approx 1$, then we have $y'(2) \approx 2 - 1 = 1$.
- We approximate $y(3) \approx y(2) + 1 \cdot y'(2) \approx 2$, then we have $y'(3) \approx 3 - 2 = 1$.
- We approximate $y(4) \approx y(3) + 1 \cdot y'(3) \approx 3$, then we have $y'(4) \approx 4 - 3 = 1$.
- We approximate $y(5) \approx y(4) + 1 \cdot y'(4) \approx 4$.

Then, we have approximated that:
$$y(5) \approx \boxed{4}.$$

6. For the first-order autonomous ODE:

$$\frac{dx}{dt} = x^2 - 2x + c,$$

with parameter $c \in \mathbb{R}$, do the following:

(a) Sketch all of the qualitatively different graphs of $f(x) = x^2 - 2x + c$, as c is varied.

(b) Determine any and all bifurcation values for the parameter c.

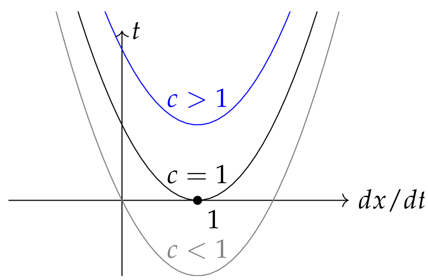
(c) Sketch a bifurcation diagram for this ODE.

Solutions:

(a) Given the right hand side, we want to find its critical point, *i.e.*, $x^2 - 2x + c = 0$, that is:

$$x = \frac{2 \pm \sqrt{4-4c}}{2} = 1 \pm \sqrt{1-c}.$$

Graphically, we may draw the diagram as:

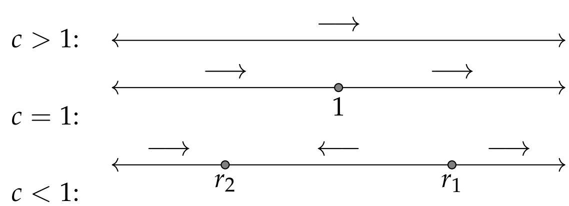


(b) Then, we find the bifurcation value, that is the critical points equivalent to each other, *i.e.*:

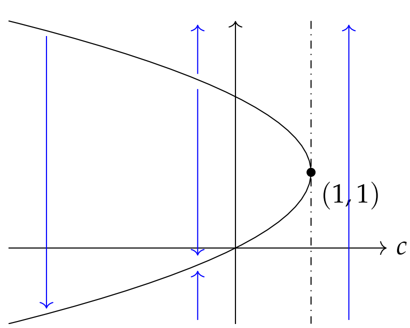
$$1 + \sqrt{1-c} = 1 - \sqrt{1-c}$$
$$c = \boxed{1}.$$

Diligent readers might also notice that this is the value c such that $\Delta = 0$.

(c) When $c > 1$, $dx/dt > 0$. When $c = 1$, the two roots are both 1. When $c < 1$, the $dx/dt > 0$ when larger than the larger root or smaller than the smaller roots, hence the phase diagrams are:



Thus, the bifurcation diagram is:



7. Carbon-14, a radioactive isotope of carbon, is an effective tool in dating the age of organic compounds, as it decays with a relatively long period. Let $Q(t)$ denote the amount of carbon-14 at time t, we suppose that the decay of $Q(t)$ satisfies the following differential equation:

$$\frac{dQ}{dt} = -\lambda Q \text{ where } \lambda \text{ is the rate of decay constant.}$$

   (a) Let the half-life of carbon-14 be τ, find the rate of decay, λ.

   (b) Suppose that a piece of remain is discovered to have 10% of the original amount of carbon-14, find the age of the remain in terms of τ.

Solutions:

(a) Note that the differential equation is separable, hence:

$$\begin{aligned} \frac{dQ}{Q} &= -\lambda dt, \\ \int \frac{dQ}{Q} &= -\int \lambda dt, \\ \log|Q| &= -\lambda t + C, \\ Q &= \tilde{C} e^{-\lambda t}. \end{aligned}$$

Here, we assume $Q = Q_0$ at $t = t_0$, then we have $Q = Q_0/2$ when $t = t_0 + \tau$, so:

$$\frac{1}{2} = e^{-\lambda\tau},$$

which deduces to:

$$\lambda = -\frac{1}{\tau}\log\left(\frac{1}{2}\right) = \boxed{\frac{\log 2}{\tau}}.$$

(b) If there are only 10% of remain, we suppose that we have $Q = Q_0$ at $t = 0$, and have $Q = Q_0/10$ at $t = t_0$, hence giving that:

$$\frac{Q_0}{10} = Q_0 \exp\left(-\lambda t_0\right) = Q_0 \exp\left(-\frac{\log(2) t_0}{\tau}\right).$$

Thus, we obtain that:

$$\frac{1}{10} = \exp\left(-\frac{\log(2) t_0}{\tau}\right),$$

and by solving for t_0, we obtain:

$$t_0 = -\frac{\tau}{\log 2}\log\left(\frac{1}{10}\right) = \boxed{\frac{\log 10}{\log 2}\tau}.$$

8.* Let an initial value problem be defined as follows:

$$\begin{cases} (12x^4 + 5x^2 + 6)\dfrac{dy}{dx} - (x^2\sin(x) + x^3)y = 0, \\ y(0) = 1. \end{cases}$$

Show that the solution to the above initial value problem is symmetric about $x = 0$.

Solution:

If you were attempting to solve this problem by integrating factor or exactness, you are on the wrong track. The functions are deliberately chosen so that these operations will be hardly possible.

However, this does not necessarily mean that is it not possible to prove without solving the solution out, one shall utilize the existence and uniqueness theorem to proceed.

Proof. Now, we first observe that when we rewrite the problem, we have:

$$y' = \frac{x^2\sin(x) + x^3}{12x^4 + 5x^2 + 6}y,$$

where we clearly notice that the numerator and denominator are composed of continuous function while the denominator is positive, so we know that it is continuous over $\mathbb{R}$, so the initial value problem exhibits a unique solution.

Now, suppose $y(x)$ is a solution of the above IVP, we want to show that $\tilde{y}(x) := y(-x)$ is also a solution to the above IVP.

Clearly, we have:

$$\tilde{y}(0) = y(-0) = y(0) = 1,$$

so the initial condition is satisfied, so we are left to check the differential equation. By chain rule, we have:

$$\frac{d\tilde{y}}{dx}(x) = \frac{dy}{dx}(-x)\cdot\frac{d}{dx}[-x] = -\tilde{y}'(x).$$

With the first equation and y being a solution, we can make all x into $-x$ to obtain that:

$$(12(-x)^4 + 5(-x)^2 + 6)\frac{dy(-x)}{dx} - ((-x)^2\sin(-x) + (-x)^3)\tilde{y} = 0,$$

and if we organize the left hand side, we have:

$$(12x^4 + 5x^2 + 6)\tilde{y}' - (x^2\sin(x) + x^3)\tilde{y} = 0,$$

so $\tilde{y}$ is clearly another solution to the IVP, so by uniqueness, we must have $\tilde{y}(x) = y(x)$, or namely $y(-x) = y(x)$, so the solution must be symmetric about $x = 0$. □

9.* The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:

$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x, 0) = \sin\left(\dfrac{5\pi x}{L}\right), & \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$

where c is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x, t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is non-trivial.

Hint: Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis.

Solution:

With the method of separation, we insert the separations back to the system of equation to obtain:

$$v(x)w''(t) = c^2 v''(x)w(t).$$

Now, we apply the separation and set the common ratio to be λ:

$$\frac{v''(x)}{v(x)} = \frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = \lambda.$$

Reformatting the boundary condition gives use the following initial value problem:

$$\begin{cases} v''(x) - \lambda v(x) = 0, \\ v(0) = v(L) = 0. \end{cases}$$

As a second order linear ordinary differential equation, we discuss all following cases:

- If $\lambda = 0$, then $v(x) = a + Bx$ and by the initial condition, $A = B = 0$, which gives the trivial solution, *i.e.*, $v(x) = 0$;
- If $\lambda = \mu^2 > 0$, then we have $v(x) = Ae^{-\mu x} + Be^{\mu x}$ and again giving that $A = B = 0$, or the trivial solution;
- Eventually, if $\lambda = -\mu^2 < 0$, then we have the solution as:

  $$v(x) = A \sin(\mu x) + B \cos(\mu x),$$

  and the initial conditions gives us that:

  $$\begin{cases} v(0) = B = 0, \\ v(L) = A \sin(\mu L) + B \cos(\mu L) = 0, \end{cases}$$

  where A is arbitrary, $B = 0$, and $\mu L = m\pi$ positive integer m.

Overall, the only non-trivial solution would be:

$$v_m(x) = A \sin(\mu_m x), \text{ where } \mu_m = \frac{m\pi}{L}.$$

Eventually, by inserting back $\lambda = -\mu_m^2$, we have $\lambda = -m^2\pi^2/L^2$, giving the solution to $w_m(t)$, another second order linear ordinary differential equation, as:

$$w_m(t) = C \cos(\mu_m ct) + D \sin(\mu_m ct), \text{ with } C, D \in \mathbb{R}.$$

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By the *principle of superposition*, we can have our solution in the form:

$$u(x,t) = \sum_{m=1}^{\infty} \left[a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)\right] \sin(\mu_m x),$$

where our coefficients a_m and b_m have to be chosen to satisfy the initial conditions for $x \in [0, L]$:

$$u(x,0) = \sum_{m=1}^{\infty} a_m \sin(\mu_m x) = \sin\left(\frac{2\pi x}{L}\right),$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin(\mu_m x) = \sin\left(\frac{5\pi x}{L}\right).$$

Since we are hinted that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis, we now know that except for the following:

$$a_2 = 1 \text{ and } c\mu_5 b_5 = 1,$$

all the other coefficients are zero, so we have:

$$u(x,t) = \boxed{\cos\left(\frac{2\pi ct}{L}\right)\sin\left(\frac{2\pi x}{L}\right) + \frac{L}{5\pi c}\sin\left(\frac{5\pi ct}{L}\right)\sin\left(\frac{5\pi x}{L}\right)}.$$

10. Given a differential equation for $y = y(t)$ being:

$$t^3 y'' + ty' - y = 0.$$

(a) Verify that $y_1(t) = t$ is a solution to the differential equation.

(b) Find the full set of solutions using reduction of order.

(c) Show that the set of solutions from part (b) is linearly independent.

Solution:

(a) *Proof.* We note that the left hand side is:

$$t^3 y_1'' + ty_1' - y_1 = t^3 \cdot 0 + t \cdot 1 - t = t - t = 0.$$

Hence $y_1(t) = t$ is a solution to the differential equation. □

(b) By reduction of order, we assume that the second solution is $y_2(t) = tu(t)$, then we plug $y_2(t)$ into the equation to get:

$$2t^3 u'(t) + t^4 u''(t) + tu(t) + t^2 u'(t) = t^4 u''(t) + (2t^3 + t^2)u'(t) = 0.$$

Here, we let $\omega(t) = u'(t)$ to get a first order differential equation:

$$t^2 \omega'(t) = (-2t - 1)\omega(t).$$

Clearly, this is separable, and we get that:

$$\frac{\omega'(t)}{\omega(t)} = -\frac{2t+1}{t^2} = -\frac{2}{t} - \frac{1}{t^2},$$

which by integration, we have obtained that:

$$\log\big(\omega(t)\big) = -2\log t + \frac{1}{t} + C.$$

By taking exponentials on both sides, we have:

$$\omega(t) = \exp\left(-2\log t + \frac{1}{t} + C\right) = \widetilde{C} e^{1/t} \cdot \frac{1}{t^2}.$$

Recall that we want $u(t)$ instead of $\omega(t)$, so we have:

$$u(t) = \int \omega(t) dt = \widetilde{C} \int e^{1/t} \cdot \frac{1}{t^2} dt = -\widetilde{C} e^{1/t} + D.$$

By multiplying t, we obtain that:

$$y_2 = -\widetilde{C} t e^{1/t} + Dt,$$

where Dt is repetitive in y_1, so we get:

$$y(t) = \boxed{C_1 t + C_2 t e^{1/t}}.$$

(c) *Proof.* We calculate Wronskian as:

$$W[t, te^{1/t}] = \det \begin{pmatrix} t & te^{1/t} \\ 1 & e^{1/t} - \frac{e^{1/t}}{t} \end{pmatrix} = -e^{1/t} \neq 0,$$

hence the set of solutions is linearly independent. □