Exam 1 Review Problem Set 2

Differential Equations

Summer 2025

Instructions:

The set of questions serves as PILOT practices to Exam 1 for the Summer 2025 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- Questions marked with an asterisk are more challenging.
- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.

1. Solve for the general solution to the following ODEs with $y = y(t)$:

   (a) $$2y' + y = 3t.$$

   (b) $$y' + \log(t)y = t^{-t}.$$

2. Solve the following initial value problem (IVP) on $y = y(x)$, and specify the domain for your solution:

$$\begin{cases} y' = y(y+1), \\ y(0) = 1. \end{cases}$$

3. Given an initial value problem:

$$\begin{cases} \dfrac{dy}{dt} - \dfrac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$

   (a) Find the integrating factor $\mu(t)$.

   (b) Solve for the particular solution for the initial value problem.

   (c) Discuss the behavior of the solution as $t \to \infty$ for different cases of y_0.

4. An autonomous differential equation is given as follows:

$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \text{ where } t \geq 0 \text{ and } y \geq 0.$$

   Draw a phase portrait and sketch a few solutions with different initial conditions.

5. Let a differential equation be defined as:

$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$

Use Euler's Method with step size $h = 1$ to approximate $y(5)$.

6. For the first-order autonomous ODE:

$$\frac{dx}{dt} = x^2 - 2x + c,$$

with parameter $c \in \mathbb{R}$, do the following:

(a) Sketch all of the qualitatively different graphs of $f(x) = x^2 - 2x + c$, as c is varied.

(b) Determine any and all bifurcation values for the parameter c.

(c) Sketch a bifurcation diagram for this ODE.

7. Carbon-14, a radioactive isotope of carbon, is an effective tool in dating the age of organic compounds, as it decays with a relatively long period. Let $Q(t)$ denote the amount of carbon-14 at time t, we suppose that the decay of $Q(t)$ satisfies the following differential equation:

$$\frac{dQ}{dt} = -\lambda Q \text{ where } \lambda \text{ is the rate of decay constant.}$$

(a) Let the half-life of carbon-14 be τ, find the rate of decay, λ.

(b) Suppose that a piece of remain is discovered to have 10% of the original amount of carbon-14, find the age of the remain in terms of τ.

8.* Let an initial value problem be defined as follows:

$$\begin{cases} (12x^4 + 5x^2 + 6)\dfrac{dy}{dx} - \left(x^2 \sin(x) + x^3\right)y = 0, \\ y(0) = 1. \end{cases}$$

Show that the solution to the above initial value problem is symmetric about $x = 0$.

9.* The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:

$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x, 0) = \sin\left(\dfrac{5\pi x}{L}\right), & \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$

where c is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x, t) =$

$v(x) \cdot w(t)$, and attempt to obtain a general solution that is non-trivial.
Hint: Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis.

10. Given a differential equation for $y = y(t)$ being:

$$t^3 y'' + ty' - y = 0.$$

(a) Verify that $y_1(t) = t$ is a solution to the differential equation.

(b) Find the full set of solutions using reduction of order.

(c) Show that the set of solutions from part (b) is linearly independent.