Exam 2 Review Problem Set 3: Solutions

Differential Equations

Summer 2025

1. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

   (a) $$y'' + 2y' + y = e^{-t}.$$

   (b) $$y'' + y = \tan t.$$

Solution:

(a) First, we look for homogeneous solution, *i.e.*, $y'' + 2y' + y = 0$, whose characteristic equation is:

$$r^2 + 2r + 1 = (r+1)^2 = 0,$$

with root(s) being -1 with multiplicity of 2, so the general solution to homogeneous case is:

$$y_g(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

Notice that the non-homogeneous part is e^{-t}, but we have e^{-t} and te^{-t} as general solutions already, so we have our guess of particular solution as:

$$y_p(t) = At^2 e^{-t}.$$

By taking the derivatives, we have:

$$y_p'(t) = A(2te^{-t} - t^2 e^{-t}) \qquad \text{and} \qquad y_p''(t) = A(2e^{-t} - 4te^{-t} + t^2 e^{t}).$$

We simply plug in the particular solution, so we have:

$$\begin{aligned} A(2e^{-t} - 4te^{-t} + t^2 e^{t}) + 2A(2te^{-t} - t^2 e^{-t}) + At^2 e^{-t} &= e^{-t} \\ 2Ae^{-t} &= e^{-t} \\ A &= \frac{1}{2}. \end{aligned}$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}}.$$

(b) Here, we still look for homogeneous solutions, *i.e.*, $y'' + y = 0$, whose characteristic equation is:

$$r^2 + 1 = 0,$$

with roots $\pm \mathrm{i}$. Since we are dealing with real valued functions, we have the general solution as:

$$y_g = C_1 \sin t + C_2 \cos t.$$

Note that $\tan t$ does not work with undetermined coefficients, we must use the variation of parameters, the Wronskian of our solution is:

$$\mathrm{W}[\sin t, \cos t] = \det \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} = -\sin^2 t - \cos^2 t = -1.$$

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Now, we may use the formula, namely getting the particular solution as:

$$\begin{aligned} y_p &= \sin t \int \frac{-\cos t \cdot \tan t}{-1} dt + \cos t \int \frac{\sin t \cdot \tan t}{-1} dt \\ &= \sin t \int \sin t dt - \cos t \int \frac{\sin^2 t}{\cos t} dt \\ &= \sin t(-\cos t + C) - \cos t \int \frac{1 - \cos^2 t}{\cos t} dt \\ &= -\sin t \cos t - \cos t \left(\log|\sec t + \tan t| - \sin t + C\right) \\ &= -\cos t \log|\sec t + \tan t|. \end{aligned}$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 \sin t + C_2 \cos t - \cos t \log|\sec t + \tan t|}.$$

2. Solve for the general solution to the following higher order ODE.

$$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

Solution:

Note that we obtain the characteristic equation as:

$$4r^4 - 24r^3 + 45r^2 - 29r + 6 = 0.$$

To obtain our roots, we use the **Rational Root Theorem**, so if the characteristic equation has any rational root, it must have been one (or more) of the following:

$$\pm 1, \pm 2, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}.$$

From plugging in the values, we notice that 2 and 3 are roots of the characteristic equation, by division, we have:

$$\frac{4r^4 - 24r^3 + 45r^2 - 29r + 6}{(r-2)(r-3)} = 4r^2 - 4r + 1 = (2r-1)^2.$$

Now, we know that the roots are 2, 3, and $1/2$ with multiplicity 2, thus the solution to the differential equation is:

$$y(x) = \boxed{C_1e^{2x} + C_2e^{3x} + C_3e^{x/2} + C_4xe^{x/2}}.$$

Again, we invite readers to verify the ***Rational Root Theorem****, which appears to be an important result from algebra.*

3.* Solve for the general solution to the following fourth order ODE.

$$\frac{d^4y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

Solution:

For this general solution, we trivially obtain that the characteristic polynomial is:

$$r^4 + 1 = 0.$$

Recall that the root of unity address for the case when $r^n = 1$, so we consider the 8th root of unity, in which $(\zeta_8)^8 = 1$. Now, recall **Euler's Identity** and **deMoivre's formula**, we note that only the odd powers of the 8th root of unity satisfies that $r^4 = -1$, namely, are:

$$\begin{aligned}
\zeta_8 &= \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}, \\
\zeta_8^3 &= \cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2}, \\
\zeta_8^5 &= \cos\left(\frac{5\pi}{4}\right) + \mathrm{i}\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}, \\
\zeta_8^7 &= \cos\left(\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}.
\end{aligned}$$

Also, we note that ζ_8 and ζ_8^7 are complex conjugates, whereas ζ_8^3 and ζ_8^5 are complex conjugates, so we can linearly combine them to obtain the set of linearly independent solutions, *i.e.*:

$$y(x) = \boxed{\begin{aligned} & e^{(\sqrt{2}/2)x}\left[C_1\cos\left(\frac{\sqrt{2}}{2}x\right) + C_2\sin\left(\frac{\sqrt{2}}{2}x\right)\right] \\ + \ & e^{-(\sqrt{2}/2)x}\left[C_3\cos\left(\frac{\sqrt{2}}{2}x\right) + C_4\sin\left(\frac{\sqrt{2}}{2}x\right)\right]\end{aligned}}.$$

4.* Recall that we have defined linear independence of functions, we define *orthogonality* of two real-valued, "square-integrable" functions over $[0, 2\pi]$, f and g, as:

$$\int_0^{2\pi} f(x)g(x)dx = 0.$$

(a) Show that the set $\{\sin x, \cos x\}$ is linearly independent and orthogonal.

(b) Show that if $\{f(x), g(x)\}$ is orthogonal, then $C_1 f(x)$ and $C_2 g(x)$ is orthogonal.

(c) Note that $\{x, x^2\}$ are linearly independent, construct a basis that is orthogonal.

In fact, this is how Fourier series is being defined. Here, $\{\sin(nx), \cos(nx)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis of $L^2([0, 2\pi])$ space.

(d) Verify that $\{\sin(nx), \cos(nx), 1\}_{n\in\mathbb{Z}^+}$ is an orthogonal set.

Note that the verification of it being a basis is, in fact, much more complicated, so we will just bear with that. However, for any function $f \in L^2([0, 2\pi])$, it is defined such that:

$$\int_0^{2\pi} \left(f(x)\right)^2 dx < +\infty.$$

(e) Verify that $f(x) = x$ is a $L^2([0, 2\pi])$ function.

(f) Decompose $f(x) = x$ into sine and cosine functions, this is a Fourier series of $f(x) = x$.

Solution:

(a) *Proof.* To show linear independence, we compute the Wronskian as:

$$\mathrm{W}[\sin x, \cos x] = \det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

Then, to show orthogonality, we have:

$$\int_0^{2\pi} \sin x \cos x dx = \frac{1}{2}\int_0^{2\pi} \sin(2x)dx = \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right]_0^{2\pi} = \frac{1}{4}\left(\cos 0 - \cos(4\pi)\right) = 0,$$

hence we have shown linear independence and orthogonality. □

(b) *Proof.* By orthogonality, we have $\int_0^{2\pi} f(x)g(x)dx = 0$, so we have:

$$\int_0^{2\pi} C_1 f(x) \cdot C_2 g(x)dx = C_1C_2 \int_0^{2\pi} f(x)g(x)dx = C_1C_2 \cdot 0 = 0.$$

Hence orthogonality is preserved with scalar multiplications. □

(c) The check of x and x^2 being linearly independent can be verified by Wronskian, and we leave this check to the readers. By the principle of superposition, we want to construct the second argument as $x^2 - Ax$, where A is a constant, now we take the inner product as:

$$\int_0^{2\pi} x(x^2 - Ax)dx = \int_0^{2\pi} (x^3 - Ax^2)dx = \frac{x^4}{4} - \frac{Ax^3}{3}\Bigg|_0^{2\pi} = 4\pi^4 - \frac{8A\pi^3}{3} = 0,$$

which forces A to be $3\pi/2$, so the orthogonal basis is now:

$$\boxed{\left\{x, x^2 - \frac{3\pi x}{2}\right\}}.$$

Diligent readers should notice that we have somehow constructed a "vector space" with a proper inner product. In fact, this space $L^2([0, 2\pi])$ is considered a Hilbert Space, that is a infinite dimensional vector space with completeness and denseness. The $L^2([0, 2\pi])$ is closely related to Fourier series, that has inarguable impacts on mathematics as well as sciences and engineering disciplines.

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(d) We just need to discuss a few cases separately.

- Case for $\sin(nx)$ and $\sin(mx)$, where $n \neq m$:
$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2}\int_0^{2\pi} [\cos((n-m)x) - \cos((n+m)x)]\,dx$$
$$= \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x) - \frac{1}{n+m}\sin((n+m)x)\right]_{x=0}^{x=2\pi} = 0.$$
- Case for $\sin(nx)$ and $\cos(mx)$:
$$\int_0^{2\pi} \sin(nx)\cos(mx)\,dx = \frac{1}{2}\int_0^{2\pi} [\sin((n+m)x) + \sin((n-m)x)]\,dx$$
$$= \frac{1}{2}\left[-\frac{1}{n+m}\cos((n+m)x) - \frac{1}{n-m}\cos((n-m)x)\right]_{x=0}^{x=2\pi} = 0.$$
- Case for $\cos(nx)$ and $\cos(mx)$, where $n \neq m$:
$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = \frac{1}{2}\int_0^{2\pi} [\cos((n-m)x) + \cos((n+m)x)]\,dx$$
$$= \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x) + \frac{1}{n+m}\sin((n+m)x)\right]_{x=0}^{x=2\pi} = 0.$$

Hence, we have verified that this is a orthogonal set. Note that if you want an orthonormal set, you can easily normalize it via:
$$\left\{\frac{\sin(nx/2)}{\sqrt{\pi}}, \frac{\cos(nx/2)}{\sqrt{\pi}}\right\}_{n\in\mathbb{Z}^+}.$$
This is because:

- Case for $\sin(nx)$ and $\sin(nx)$:
$$\int_0^{2\pi} \sin(nx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi} [1-\cos(2nx)]\,dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin(2nx)\right]_{x=0}^{x=2\pi} = \pi.$$
- Case for $\cos(nx)$ and $\cos(nx)$:
$$\int_0^{2\pi} \cos(nx)\cos(nx)\,dx = \frac{1}{2}\int_0^{2\pi} [1+\cos(2nx)]\,dx = \frac{1}{2}\left[x + \frac{1}{2n}\sin(2nx)\right]_{x=0}^{x=2\pi} = \pi.$$

In fact, one can prove that this is in fact a basis, *i.e.*, it spans the whole $L^2([0,2\pi])$ space. There have been proofs by using Poisson kernel, complex analysis, or by the convergence of Fourier series. We will leave this as an interests for who might get interested in this.

(e) The verification that $f(x) = x$ is trivial:
$$\int_0^{2\pi} (f(x))^2 dx = \int_0^{2\pi} x^2 dx = \frac{1}{3}x^3\Big|_{x=0}^{x=2\pi} = \frac{1}{3}\cdot 8\pi = \frac{8}{3}\pi < +\infty.$$
Hence, we have verified that $f(x) = x$ is $L^2([0,2\pi])$.

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(f) Note that we have the orthogonal basis, so we can think about project $f(x) = x$ to each basis:

- Project x to $\sin(nx)$ for $n \in \mathbb{Z}^+$:

$$\begin{aligned}
\text{proj}_{\sin(nx)}(x) &= \frac{\langle \sin(nx), x\rangle}{\langle \sin(nx), \sin(nx)\rangle} \sin(nx) \\
&= \frac{\int_0^{2\pi} x\sin(nx)dx}{\pi} \sin(nx) \\
&= \left[-\frac{1}{n}x\cos(nx) + \frac{1}{n}\int \cos(nx)\,dx\right]_{x=0}^{x=2\pi} \frac{\sin(nx)}{\pi} \\
&= \left[-\frac{2\pi}{n}\cos(2n\pi) + 0\right] \cdot \frac{1}{\pi}\sin(nx) = -\frac{2}{n}\sin(nx).
\end{aligned}$$

- Project x to $\cos(nx)$ for $n \in \mathbb{Z}^+$:

$$\begin{aligned}
\text{proj}_{\cos(nx)}(x) &= \frac{\langle \cos(nx), x\rangle}{\langle \cos(nx), \cos(nx)\rangle} \cos(nx) \\
&= \frac{\int_0^{2\pi} x\cos(nx)dx}{\pi} \cos(nx) \\
&= \left[\frac{1}{n}x\sin(nx) - \frac{1}{n}\int \sin(nx)\,dx\right]_{x=0}^{x=2\pi} \frac{\cos(nx)}{\pi} = 0.
\end{aligned}$$

- Project x to 1:

$$\text{proj}_1(x) = \frac{\langle 1, x\rangle}{\langle 1, 1\rangle} \cdot 1 = \frac{\int_0^{2\pi} xdx}{\int_0^{2\pi} 1dx} = \frac{\frac{x^2}{2}\Big|_{x=0}^{x=2\pi}}{2\pi} = \pi.$$

Hence, the Fourier series of x on $[0, 2\pi]$ is:

$$f(x) = \boxed{\pi - \sum_{n=1}^{\infty} \frac{2}{n}\sin(nx)}.$$

5. Consider the following initial value problem:

$$\begin{cases} 2y''' - 11y'' + 17y' - 6y = 0, \\ y(0) = 3, y\big(\log(4)\big) = 82, y\big(\log(9)\big) = 813. \end{cases}$$

Find the specific solution to the IVP.

Solution:

Here, we first find the characteristic equation, that is:

$$2r^3 - 11r^2 + 17r - 6 = 0.$$

Here, we would want to find the roots of the characteristic polynomial. By the rational root test, we know that if the polynomial has a rational root, it must be one of $\pm 6/\pm 2$, we may test that $r = 2$ is a root, so we can make a division to obtain $(2r^3 - 11r^2 + 17r - 6)/(r-2) = 2r^2 - 7r + 3 = (2r-1)(r-3)$, so the roots are $1/2$, 2, and 3, so the general solution is:

$$y = C_1 e^{t/2} + C_2 e^{2t} + C_3 e^{3t}, \text{ where } C_1, C_2, C_3 \text{ are constants.}$$

Then, we need to apply to the initial conditions:

$$\begin{cases} C_1 + C_2 + C_3 = 3, \\ 2C_1 + 16C_2 + 64C_3 = 82, \\ 3C_1 + 81C_2 + 729C_3 = 813. \end{cases}$$

Here, we may observe that the right hand side is exactly the sum of the coefficients, hence we can have a solution as $C_1 = C_2 = C_3 = 1$. One can also use the row reduction to reduce to this solution, so the specific solution to the IVP is:

$$y = \boxed{e^{t/2} + e^{2t} + e^{3t}}.$$

6. Find the general solution to the following differential equation, and verify that your solution is a linearly independent set of solutions.

$$y'''(x) - 6y''(x) + 11y'(x) - 6y(x) = 0.$$

Solution: As usual, we first find the characteristic equation, that is:

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Up to this point, readers should be quite familiar with the *rational root theorem*, so we know that if the polynomial has a rational root, it must be one of the following:

$$\pm 1, \pm 2, \pm 3, \text{ and } \pm 6.$$

In fact, for degree 3 polynomials of integer/rational coefficients, it must have at least one rational root. *We leave the check of this claim to the readers, as an exercise to get more familiar with polynomials.*
By easy checking, we note that 1 is a root, as $1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0$, then we can eliminate the polynomial to:

$$r^3 - 6r^2 + 11r - 6 = (r-1)(r^2 - 5r + 6) = (r-1)(r-2)(r-3),$$

hence the roots are $r = 1, 2, 3$, each with multiplicity 1.
Therefore, the general solution should be:

$$y(x) = \boxed{C_1 e^x + C_2 e^{2x} + C_3 e^{3x}}.$$

Note that since we are asked to verify linear independence, we use the Wronskian, that is:

$$W[e^x, e^{2x}, e^{3x}] = \det \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix} = 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x} = 2e^{6x} \neq 0,$$

hence the set $\{e^x, e^{2x}, e^{3x}\}$ is linearly independent.

7. Find the solution of $y = y(t)$ to the following IVP using Laplace transformation:

$$\begin{cases} y'' - 2y' + 2y = e^{-t}, \\ y(0) = 0, \qquad y'(0) = 1. \end{cases}$$

Solution:

Here, we want to use the Laplace transformation for derivatives, for simplicity, we denote $Y = \mathcal{L}\{y\}$:

$$\begin{aligned} \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \mathcal{L}\{e^{-t}\} \\ s^2Y - sy(0) - y'(0) - 2sY + 2y(0) + 2Y &= \frac{1}{s+1} \\ (s^2 - 2s + 1)Y &= \frac{1}{s+1} + 1 = \frac{s+2}{s+1} \\ Y &= \frac{s+2}{(s+1)(s^2-2s+2)} \\ &= \frac{1}{5}\frac{1}{s+1} + \frac{1}{5}\frac{-s+8}{(s-1)^2+1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}. \end{aligned}$$

By taking the inverse of Laplace, we have:

$$y(t) = \boxed{\frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos t + \frac{7}{5}e^{t}\sin t}.$$

8.* Dirac delta function $\delta(t)$ is heuristically defined as:

$$\delta(t) = \begin{cases} +\infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

In *real analysis*, $\delta(t)$ is often called an "approximation to identity", meaning that it "preserves" the original equation after convolution. By the definition of convolution for f and g, here, as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau,$$

prove that $(f * \delta)(t) = f(t)$ for $t \geq 0$.

Hint: Use the convolution theorem and the Laplace transformation of step functions.

Solution:

If you have consulted with a few other texts, you might observe that the convolution formula here is different from the convolution formula in analysis textbooks, *i.e.*:

$$(f * g)(t) = \int_0^{\infty} f(\tau)g(t-\tau)d\tau.$$

In fact, the above definition allows better property, known as "approximation to identity" for the entire domain. However, we may show the version that you see in the ODEs course with a weaker conclusion.

Proof. Here, we apply the Laplace transformation on $f * \delta$, which is:

$$\begin{aligned} \mathcal{L}\{(f * \delta)(t)\} &= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{\delta(t)\} \\ &= \mathcal{L}\{f(t)\} \cdot e^{-0s} \\ &= \mathcal{L}\{u(t) \cdot f(t)\}. \end{aligned}$$

(Or you can consider the e^{-0s} term disappearing since it is just 1.) Hence, we have demonstrated that:

$$(f * \delta)(t) = \begin{cases} 0, & \text{when } t < 0 \\ f(t), & \text{when } t \geq 0 \end{cases}$$

which implies that $(f * \delta)(t) = f(t)$ for $t \geq 0$. □

If you are interested in the concept of Dirac delta function, you can look up the conditions to be a "good kernel", which proceeds further to a narrower class of kernels as approximations to the identity. Moreover, we suggest consulting some constructions of the kernels, such as the *shrinking function* or the series of *Gaussian bell curve*.

9. Let a system of differential equations be defined as follows, find the general solutions to the equation:

$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^2.$$

Solution:
Here, we notice that the liner system is diagonal, so we can simply solve for each entry as $x_1 = e^{3t}$ and $x_2 = e^{2t}$. Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

10. Let $\mathbf{x} \in \mathbb{R}^4$, find the general solution of $\mathbf{x}$ for:

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

Solution:

For this question, readers should observe that we really have two 2-by-2 matrices and on the diagonal and zeros on the other entries.

By the definition of the eigenvalues and eigenvectors, if λ and (v_1, v_2) is the eigenvalue and eigenvector associated to the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, it should be clear that λ and $(v_1, v_2, 0, 0)$ is the eigenvalue and eigenvector associated to the full matrix. This should also apply to the lower right submatrix $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$, hence, we only need to find the eigenvalues and eigenvectors associated to the two 2-by-2 matrices:

- Consider $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, we compute $\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$, so we have $(2-\lambda)^2 - 1 = 0$, and $\lambda = 2 \pm 1$, so $\lambda_1 = 3$ and $\lambda_2 = 1$. Then, for the eigenvectors, we respectively have $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \boldsymbol{\xi_1} = \mathbf{0}$, so $\boldsymbol{\xi_1} = (1, 1)$; for $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \boldsymbol{\xi_2} = \mathbf{0}$, so $\boldsymbol{\xi_2} = (1, -1)$.
- Consider $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$, we compute $\det \begin{pmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{pmatrix} = 0$, so we have $(2-\lambda)(1-\lambda) - 12 = 0$, which is $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$, and so $\lambda_3 = 5$ and $\lambda_4 = -2$. Then, for the eigenvectors, we respectively have $\begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} \cdot \boldsymbol{\xi_3} = \mathbf{0}$, so $\boldsymbol{\xi_3} = (4, 3)$; for $\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \cdot \boldsymbol{\xi_4} = \mathbf{0}$, so $\boldsymbol{\xi_4} = (1, -1)$.

Hence we have the solution as:

$$\mathbf{x} = \boxed{C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{5t} \begin{pmatrix} 0 \\ 0 \\ 4 \\ 3 \end{pmatrix} + C_4 e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}}.$$