PILOT

Exam 2 Review Problem Set 3

Differential Equations

Summer 2025

Instructions:

The set of questions serves as PILOT practices to Exam 2 for the Summer 2025 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- Questions marked with an asterisk are more challenging.
- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.

1. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

   (a) $$y'' + 2y' + y = e^{-t}.$$

   (b) $$y'' + y = \tan t.$$

2. Solve for the general solution to the following higher order ODE.

$$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

3.* Solve for the general solution to the following fourth order ODE.

$$\frac{d^4y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

4.* Recall that we have defined linear independence of functions, we define *orthogonality* of two real-valued, "square-integrable" functions over $[0, 2\pi]$, f and g, as:

$$\int_0^{2\pi} f(x)g(x)dx = 0.$$

(a) Show that the set $\{\sin x, \cos x\}$ is linearly independent and orthogonal.

(b) Show that if $\{f(x), g(x)\}$ is orthogonal, then $C_1 f(x)$ and $C_2 g(x)$ is orthogonal.

(c) Note that $\{x, x^2\}$ are linearly independent, construct a basis that is orthogonal.

In fact, this is how Fourier series is being defined. Here, $\{\sin(nx), \cos(nx)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis of $L^2([0, 2\pi])$ space.

(d) Verify that $\{\sin(nx), \cos(nx), 1\}_{n\in\mathbb{Z}^+}$ is an orthogonal set.

Note that the verification of it being a basis is, in fact, much more complicated, so we will just bear with that. However, for any function $f \in L^2([0, 2\pi])$, it is defined such that:

$$\int_0^{2\pi} \left(f(x)\right)^2 dx < +\infty.$$

(e) Verify that $f(x) = x$ is a $L^2([0, 2\pi])$ function.

(f) Decompose $f(x) = x$ into sine and cosine functions, this is a Fourier series of $f(x) = x$.

5. Consider the following initial value problem:

$$\begin{cases} 2y''' - 11y'' + 17y' - 6y = 0, \\ y(0) = 3, y\big(\log(4)\big) = 82, y\big(\log(9)\big) = 813. \end{cases}$$

Find the specific solution to the IVP.

6. Find the general solution to the following differential equation, and verify that your solution is a linearly independent set of solutions.

$$y'''(x) - 6y''(x) + 11y'(x) - 6y(x) = 0.$$

7. Find the solution of $y = y(t)$ to the following IVP using Laplace transformation:

$$\begin{cases} y'' - 2y' + 2y = e^{-t}, \\ y(0) = 0, \qquad y'(0) = 1. \end{cases}$$

8.* Dirac delta function $\delta(t)$ is heuristically defined as:

$$\delta(t) = \begin{cases} +\infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

In *real analysis*, $\delta(t)$ is often called an "approximation to identity", meaning that it "preserves" the original equation after convolution. By the definition of convolution for f and g, here, as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau,$$

prove that $(f * \delta)(t) = f(t)$ for $t \geq 0$.
Hint: Use the convolution theorem and the Laplace transformation of step functions.

9. Let a system of differential equations be defined as follows, find the general solutions to the equation:

$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^2.$$

10. Let $\mathbf{x} \in \mathbb{R}^4$, find the general solution of $\mathbf{x}$ for:

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \mathbf{x}.$$