PILOT

Exam 2 Review Problem Set 4: Solutions

Differential Equations

Summer 2025

1. Solve the following differential equations.

   (a) $$y'' + 4y = t^2 + 3e^t.$$

   (b) $$y'' + 2y' + y = \frac{e^{-x}}{x}.$$

Solution:

(a) For the first part, we first find the solution to the homogeneous case, that is $y'' + 4y = 0$, whose characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm 2\mathrm{i}$. Hence, the homogeneous solution is:

$$y = C_1 \cos(2t) + C_2 \sin(2t).$$

Based on the non-homogeneous part, our guess of the solution should be:

$$y_p(t) = \underbrace{At^2 + Bt + C}_{\text{Guess for } t^2} + \underbrace{De^t}_{\text{Guess for } 3e^t}.$$

Of course, readers can make separated guess since differentiation is linear operator, and solve for a, b, c and d separately. However, we will provide the whole derivatives as:

$$y_p' = 2At + B + De^t, \qquad y_p'' = 2A + De^t.$$

Therefore, as we plug in the particular solution, we have:

$$(2A + De^t) + 4(At^2 + Bt + C + De^t) = 4At^2 + Bt + (4C + 2A) + 5De^t = t^2 + 3e^t,$$

so the solutions are $A = 1/4$, $B = 0$, $C = -1/8$, and $D = 3/5$, so we have:

$$y(t) = \boxed{C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t}.$$

(b) Still, we first look for the homogeneous solution, for $y'' + 2y' + y = 0$, with characteristic equation as $r^2 + 2r + 1 = 0$, the roots is $r = -1$ with multiplicity 2, that is $y = C_1 e^{-x} + C_2 x e^{-x}$. Here, we use the variation of parameter that we first take the Wronskian:

$$\mathrm{W}[e^{-x}, xe^{-x}] = \det \begin{pmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{pmatrix} = -xe^{2x} + e^{-2x} + xe^{-2x} = e^{-2x}.$$

Therefore, we have the particular solution as:

$$\begin{aligned} y_p(x) &= -y_1(x) \int \frac{y_2(x) g(x)}{\mathrm{W}} dx + y_2(x) \int \frac{y_1(x) g(x)}{\mathrm{W}} dx \\ &= -e^{-x} \int \frac{xe^{-x}}{e^{-2x}} \frac{e^{-x}}{x} dx + xe^{-x} \int \frac{e^{-x}}{e^{-2x}} \frac{e^{-x}}{x} dx \\ &= -e^{-x} \int dx + xe^{-x} \int \frac{dx}{x} = -xe^{-x} + K_1 e^{-x} + K_2 x e^{-x} + xe^{-x} \log|x| \\ &= xe^{-x} \log|x|. \end{aligned}$$

Hence, the solution would be:

$$y(x) = \boxed{C_1 e^{-x} + C_2 x e^{-x} + xe^{-x} \log|x|}.$$

2.* Find a full set of real solutions to the differential equation:

$$\frac{d^3y}{dx^3} = -y.$$

Solution:

Clearly, the characteristic equation is $r^3 = -1$. For this part, you will still have two options to proceed:

- By observing that -1 is a result, you may induct a long division of $(r^3+1)/(r+1)$, and factor as of how you factor quadratics, or
- by Euler's method's heuristics, namely finding the roots for $x^6 = 1$, that is $\zeta_6 = e^{2\pi i/6} = e^{\pi i/3}$, and take its odd powers, that is ζ_6^1, ζ_6^3, and ζ_6^5.

Whatever your choice is, you should obtain your three roots as:

$$r = -1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \text{ and } \frac{1}{2} - i\frac{\sqrt{3}}{2},$$

hence inducing the solution set as:

$$\left\{e^{-t}, e^{(1/2+i\sqrt{3}/2)t}, e^{(1/2-i\sqrt{3}/2)t}\right\}.$$

By some simply arithmetics of linear combinations, we have:

$$\boxed{\left\{e^{-t}, e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right), e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right\}}.$$

At this moment, readers should be utterly clear with Euler's identity and the method of transforming from complex-valued solutions to real-valued solution. If you are having trouble on this question, we suggest you to review on this part and check on the previous solutions.

3. Let a differential equation of $y := y(x)$ be:

$$y''' + 3y'' + 3y' + y = 0.$$

Find the general solution the differential equation and give the Wronskian of your set of solutions.

Solution:

Here, the characteristic equation is:

$$r^3 + 3r^2 + 3r + 1 = (r+1)^3,$$

so the root is $r = -1$ with multiplicity 3.

Hence, the general solution is:

$$y(x) = \boxed{C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}}.$$

Then, we compute the Wronskian as:

$$\begin{aligned} W[e^{-x}, xe^{-x}, x^2e^{-x}] &= \det \begin{pmatrix} e^{-x} & xe^{-x} & x^2e^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} & -x^2e^{-x} + 2xe^{-x} \\ e^{-x} & xe^{-x} - 2e^{-x} & x^2e^{-x} - 4xe^{-x} + 2e^{-x} \end{pmatrix} \\ &= e^{-3x}(-x^3 - 5x^2 - 6x + 2 - x^3 + 2x^2 - x^3 \\ &\quad + 2x^2 + x^3 - x^2 + x^3 - 4x^2 + 2x + x^3 - 4x^2 + 4x) \\ &= \boxed{2e^{-3x}}. \end{aligned}$$

Note that the computation should be utilizing the product rule multiple times, and it should be a good practice of computing the determinant.

4.* In our study of differential equations, our main focus is on *real-valued functions*. However, **Euler's theorem** bridges between real values and complex values.

(a) Express $\sin(z)$ and $\cos(z)$ in terms of exponential functions, where $z \in \mathbb{C}$ is a complex number.

(b) Given a function $\varphi\colon \mathbb{R} \to \mathbb{C}$ defined as $\varphi(x) = \exp(\mathrm{i}x)$. We can decompose $\varphi = i_f \circ \tilde{\varphi} \circ \pi_\sim$, where $\pi_\sim$ is surjective, i_φ is injective, and $\tilde{\varphi}$ is bijective, which can be expressed as follows:

$$\mathbb{R} \xrightarrow{\pi_\sim} X \xrightarrow[\tilde{\varphi}]{\sim} Y \xrightarrow{i_\varphi} \mathbb{C}, \qquad \varphi\colon \mathbb{R} \to \mathbb{C}$$

Find X and Y in the above commutative diagram.
Hint: Consider $\pi_\sim$ as a projection to an equivalent class, $\tilde{\varphi}$ as a modification of φ, and i_φ as a map from the image to the co-domain.

Solution:

(a) Here, readers shall recall that sine is odd and cosine is even. Now, we shall utilize this property with the exponentials, namely:

$$\exp(-\mathrm{i}z) = \cos(-z) + \mathrm{i}\sin(-z) = \cos(z) - \mathrm{i}\sin(z).$$

Combining this with the $\exp(\mathrm{i}z)$ expression, we may find that:

$$\sin(z) = \boxed{\frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}} \text{ and } \cos(z) = \boxed{\frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2}}.$$

(b) Here, lets discuss the motivation of this first, we want to decompose our function of several pieces, in which each piece has some unique properties, note that from Euler's formula, we see repetitive patterns in every 2π rotation, so our intention is to quotient out this triviality, so we have:

$$X = \boxed{\mathbb{R}/2\pi} := \big\{\{x + 2k\pi : k \in \mathbb{Z}\} : x \in \mathbb{R}\big\},$$

while the $\pi_\sim$ map will be a simple projection that projects $\mathbb{R}$ to its equivalent classes in $\mathbb{R}/2\pi$.
Since i_φ is injective, so we want Y to be the image of φ, so:

$$Y = \boxed{\mathbb{D}_1(0)} := \{x \in \mathbb{C} : |x| = 1\},$$

with the corresponding i_φ being the identity map.
Eventually, we now see $\tilde{\varphi}$ is bijective, since it is mapped to the image of φ and the injective part is enforced by having $\exp(z)$ and $\exp(z')$ being unique up to $z \sim z'$ in the equivalent class.
If you have taken abstract algebra or basic category theory, this is called a canonical decomposition, so we can think of more properties of each respective map and use various universal properties. Please check on some of these text to learn more about such decomposition.

5. Let a third order differential equation be as follows:

$$\ell\big[y(t)\big] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

Let $\ell\big[y(t)\big] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell\big[y(t)\big]$ is non-trivial.

(b) Find the particular solution to $\ell\big[y(t)\big] = \sin t$.

(c) Find the particular solution to $\ell\big[y(t)\big] = e^{-t}$.

(d)* Suppose that $\ell\big[y_1(t)\big] = f(t)$ and $\ell\big[y_2(t)\big] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\big[y_3(t)\big] = f(t) + g(t)$.

Solution:

(a) Note that the characteristic polynomial can be factorized as perfect cubes:

$$r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0,$$

its roots are $r = -1$ with multiplicity 3, so the general solution is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}}.$$

Here, the readers are invited to check, by ***Wronskian****, that set of solutions are linearly independent.*

(b) First, we want to make our guess of particular solution as:

$$y_p(t) = A\sin t + B\cos t,$$

and by taking the derivatives, we have:

$$y_p'(t) = A\cos t - B\sin t, \qquad y_p''(t) = -A\sin t - B\cos t, \text{ and } \qquad y_p'''(t) = -A\cos t + B\sin t.$$

Then, we want to plug in the results into the equation, so:

$$\begin{aligned}\ell\big[y_p(t)\big] &= (-A\cos t + B\sin t) + 3(-A\sin t - B\cos t) + 3(A\cos t - B\sin t) + A\sin t + B\cos t \\ &= (B - 3A - 3B + A)\sin t + (-A - 3B + 3A + B)\cos t \\ &= (-2A - 2B)\sin t + (2A - 2B)\cos t.\end{aligned}$$

Therefore, we can obtain the system that:

$$\begin{cases} -2A - 2B = 1, \\ 2A - 2B = 0, \end{cases}$$

which reduces to $A = -1/4$ and $B = -1/4$, so the solution is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t} - \frac{1}{4}\sin t - \frac{1}{4}\cos t}.$$

(c) Here, note that e^{-t}, te^{-t}, and t^2e^{-t} are the solutions to homogeneous case, our guess, then, is:

$$y_p(t) = At^3 e^{-t},$$

and by taking the derivatives, we have:

$$y_p'(t) = 3At^2e^{-t} - At^3e^{-t}, \qquad y_p''(t) = 6Ate^{-t} - 6At^2e^{-t} + At^3e^{-t}, \qquad \text{and}$$
$$y_p'''(t) = 6Ae^{-t} - 18Ate^{-t} + 9At^2e^{-t} - At^3e^{-t}.$$

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When we plug the derivatives back to the solutions, we note that:

$$\begin{aligned}\ell[y_p(t)] &= (6Ae^{-t} - 18Ate^{-t} + 9At^2e^{-t} - At^3e^{-t}) \\ &\quad + 3(6Ate^{-t} - 6At^2e^{-t} + At^3e^{-t}) + 3(3At^2e^{-t} - At^3e^{-t}) + (At^3e^{-t}) \\ &= 6Ae^{-t},\end{aligned}$$

which reduces to $A = 1/6$, so the solution is:

$$y(t) = \boxed{C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + \frac{1}{6}t^3e^{-t}}.$$

(d) *Proof.* Here, one should note that the derivative operator is linear, so we have that:

$$\begin{aligned}\ell[y_1(t) + y_2(t)] &= \frac{d^3}{dt^3}[y_1(t) + y_2(t)] + 3\frac{d^2}{dt^2}[y_1(t) + y_2(t)] + 3\frac{d}{dt}[y_1(t) + y_2(t)] + [y_1(t) + y_2(t)] \\ &= y_1'''(t) + 3y_1''(t) + 3y_1'(t) + y_1(t) + y_2'''(t) + 3y_2''(t) + 3y_2'(t) + y_2(t) \\ &= f(t) + g(t),\end{aligned}$$

as desired. □

6. Show the following Laplace transformation by definition.

(a)
$$\mathcal{L}\{\sin(at)\} = \frac{a}{a^2+s^2}.$$

(b)*
$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.$$

Solution:

(a) *Proof.* Here, we do the Laplace transformation via definition:

$$\begin{aligned}\mathcal{L}\{\sin(at)\} &= \int_0^\infty e^{-st}\sin(at)dt \\ &= -\frac{1}{s}e^{-st}\sin(at)\Big|_{t=0}^{t=\infty} + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)dt \\ &= \frac{a}{s}\left[-\frac{1}{s}e^{-st}\cos(at)\Big|_{t=0}^{t=\infty} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt\right] \\ &= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin(at)\}.\end{aligned}$$

Thus, we have the Laplace transformation as:

$$\mathcal{L}\{\sin(at)\} = \frac{a/s^2}{a^2/s^2+1} = \frac{a}{s^2+a^2},$$

as desired. □

(b) *Proof.* Recall that the convolution notation in this course is that:

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau,$$

so we can apply the Laplace transformation function. For simplicity, we assume that f and g behaves well enough, *i.e.*, they satisfies the conditions for Fubinii's Theorem, thus:

$$\begin{aligned}\mathcal{L}\{(f*g)(t)\} &= \int_0^\infty e^{-st}(f*g)(t)dt \\ &= \int_0^\infty e^{-st}\int_0^\infty f(\tau)g(t-\tau)d\tau dt \\ &= \int_0^\infty f(\tau)\int_0^\infty e^{-st}g(t-\tau)dtd\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau}\int_0^\infty e^{-s(t-\tau)}g(t-\tau)dtd\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau}d\tau\cdot\int_0^\infty e^{-s(t-\tau)}g(t-\tau)d(t-\tau) \\ &= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\},\end{aligned}$$

as desired. □

7. Given the following the results after Laplace transformation $F(s) = \mathcal{L}\{f(t)\}$, find each $f(t)$ prior to the Laplace transformation.

(a) $$F(s) = \frac{2s^2+4}{s^3+4s}.$$

(b)* $$F(s) = \frac{s^2}{s^2+9} - 1.$$

Solution:

(a) For this equation, one should notice that we can factor our denominator here and use partial factions, as:

$$F(s) = \frac{2s^2+4}{s(s^2+4)} = \frac{1}{s} + \frac{s}{s^2+4}.$$

By finding the inverse, we have:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \boxed{1+\cos(2t)}.$$

(b) Here, notice that we can combine the -1 into the function, as:

$$F(s) = \frac{s^2}{s^2+9} - \frac{s^2+9}{s^2+9} = \frac{-9}{s^2+9}.$$

Note that the Laplace transformation is linear, so does its inverse, so we have:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-9}{s^2+9}\right\} = -3\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \boxed{-3\sin(3t)}.$$

8. Let $\mathbf{x} \in \mathbb{R}^2$, find the general solution of $\mathbf{x}$ for:

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

Solution:

Here, we find the characteristic equation as:

$$0 = \det \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - (-1)\cdot 3 = 8 - 6\lambda + \lambda^2 = (\lambda - 2)(\lambda - 4).$$

Hence, the eigenvalues are 2 and 4, and the eigenvalues, respectively, are:

(a) For $\lambda_1 = \boxed{2}$, we have $A - 2\,\mathrm{Id}$ as $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(1)}$ such that $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, that is $3\xi_1^{(1)} - \xi_2^{(1)} = 0$, so we have $\xi_2^{(1)} = 3\xi_1^{(1)}$, so we have $\boldsymbol{\xi}^{(1)} = \boxed{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$.

(b) For $\lambda_2 = \boxed{4}$, we have $A - 4\,\mathrm{Id}$ as $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(2)}$ such that $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, that is $\xi_1^{(2)} - \xi_2^{(2)} = 0$, so we have $\xi_2^{(2)} = \xi_1^{(2)}$, so we have $\boldsymbol{\xi}^{(2)} = \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

Hence, the solution to the linear system is:

$$\mathbf{x} = \boxed{C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}.$$

9. Let $\mathbf{x} = (x_1, x_2)$ satisfy the following differential equation.
$$\mathbf{x}' = \begin{pmatrix} \frac{1}{42} & \frac{1}{21} \\ \frac{1}{14} & \frac{1}{21} \end{pmatrix} \cdot \mathbf{x}.$$

Hint: Think about the *geometric* interpretation of eigenvalues and eigenvectors and try to *simplify* the matrix. (*Otherwise, the computation is hard.*)

Solution:
In the mean time, the audiences should complain about fractions inside the matrix. But, they should quickly notice that for a linear operator L, assume it has an eigenvalue λ and an eigenvector $\boldsymbol{\xi}$, we must have:
$$L(\boldsymbol{\xi}) = \lambda \cdot \boldsymbol{\xi}.$$
Note that if we factor out a nonzero common factor k outside L, that is $\tilde{L} = \frac{1}{k}L$, we should have $\tilde{L}() = \frac{\lambda}{k} \cdot \boldsymbol{\xi}$. Hence, why don't we wrote the matrix differently?
$$\mathbf{x}' = \frac{1}{42}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \cdot \mathbf{x}.$$
Here, we find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, and the eigenvalues will be scaled by 42 while the eigenvectors are identical. Now:
$$0 = \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1).$$
Hence, the eigenvectors are:

- For $\lambda_1 = 4$: we find $\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \mathbf{0}$, so the eigenvector is $(2, 3)$.
- For $\lambda_2 = -1$: we find $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \cdot \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} = \mathbf{0}$, so the eigenvector is $(1, -1)$.

Therefore, the eigenvalues and eigenvectors for the original matrix is:
$$\lambda_1 = \frac{4}{42} = \frac{2}{21}, \boldsymbol{\xi}^{(1)} = (2, 3), \qquad \lambda_2 = -\frac{1}{42}, \boldsymbol{\xi}^{(2)} = (1, -1).$$
Therefore, we obtain the solution that:
$$\mathbf{x} = \boxed{C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t/21} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t/42}}.$$

10.* (*Putnam 2023.*) Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

- $f(0) > 0$,
- $g(0) = 0$,
- $|f'(x)| \leq |g(x)|$ for all x,
- $|g'(x)| \leq |f(x)|$ for all x, and
- $f(r) = 0$.

You may give an answer without a rigorous proof, as the proof is out of scope of the course.
Hint: Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.

Solution:
Here, we first provide a "simplified" case, *i.e.*, we are constructing a dynamical system in which we pick equality for the inequality, that is:

$$\begin{cases} |f'(x)| = |g(x)|, \text{ and} \\ |g'(x)| = |f(x)|. \end{cases}$$

Without loss of generality, we may assume that f and g are non-negative before r, so the system becomes:

$$\begin{cases} f' = -g \\ g' = f \end{cases},$$

or equivalently, $\mathrm{y} = \begin{pmatrix} f \\ g \end{pmatrix}$ that $\mathrm{y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathrm{y}$. Clearly, we observe the eigenvalues are $\pm \mathrm{i}$ as the polynomial is $\lambda^2 + 1 = 0$. Moreover, the eigenvectors for $\lambda_1 = \mathrm{i}$ is when $\begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$, in which we have $\boldsymbol{\xi} = y \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$, and that solution is:

$$\mathrm{y} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} e^{ix} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} (\cos x + \mathrm{i} \sin x) = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + \mathrm{i} \begin{pmatrix} \cos x \\ \sin x, \end{pmatrix}$$

and by conjugation, the solution should be:

$$\begin{pmatrix} f \\ g \end{pmatrix} = C_1 \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + C_2 \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

Note that with the given initial condition that $g(0) = 0$, this enforces $C_1 = 0$, thus $f(x) = C \cos x$ and $g(x) = C \sin x$, and we know that $f(r)$ is zero first at $r = \boxed{\pi/2}$.

The above version has some reasoning, but is not a rigorous proof at all, since this does not consider if r could be smaller than $\pi/2$. For students with interests, we provide the complete proof from the Putnam competition from Victor Lie, as follows.

Proof. Without loss of generality, we assume $f(x) > 0$ for all $x \in [0, r)$ as it is the first positive zero. By the fundamental theorem of calculus, we have:

$$|f'(x)| \leq |g(x)| \leq \left| \int_0^x g(s) ds \right| \leq \int_0^x |g(s)| ds \leq \int_0^t |f(s)| ds.$$

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Now, as we denote $F(x) = \int_0^x f(s)ds$, we have:

$$f'(x) + F(x) \geq 0 \text{ for } x \in [0, r].$$

For the sake of contradiction, we suppose $r < \pi/2$, then we have:

$$f'(x) \cos x + F(x) \cos x \geq 0 \text{ for } x \in [0, r].$$

Notice that the left hand side is the derivative of $f(x) \cos x + F(x) \sin x$, so an integration on $[y, r]$ gives:

$$F(r) \sin r \geq f(y) \cos y + F(y) \sin(y).$$

With some rearranging, we can have:

$$F(r) \sin r \sec^2 y \geq f(y) \sec y + F(y) \sin y \sec^2 y$$

Again, we integrate both sides with respect to y on $[0, r]$, which gives:

$$F(r) \sin^2 r \geq F(r),$$

and this is impossible, so we have a contradiction.
Hence we must have $r \geq \pi/2$, and since we have noted the solution $f(x) = C \cos x$ and $g(x) = C \sin x$, we have proven that $r = \pi/2$ is the smallest case. □