

Exam 2 Review Problem Set 4

Differential Equations Summer 2025

Instructions:

The set of questions serves as PILOT practices to Exam 2 for the Summer 2025 term of AS.110.302 Differential Equations and Applications at Johns Hopkins University.

- · Questions marked with an asterisk are more challenging.
- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.
- 1. Solve the following differential equations.

(a)
$$y'' + 4y = t^2 + 3e^t.$$

(b)
$$y'' + 2y' + y = \frac{e^{-x}}{x}.$$

2.* Find a full set of real solutions to the differential equation:

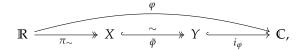
$$\frac{d^3y}{dx^3} = -y.$$

3. Let a differential equation of y := y(x) be:

$$y''' + 3y'' + 3y' + y = 0.$$

Find the general solution the differential equation and give the Wronskian of your set of solutions.

- 4.* In our study of differential equations, our main focus is on *real-valued functions*. However, **Euler's theorem** bridges between real values and complex values.
 - (a) Express $\sin(z)$ and $\cos(z)$ in terms of exponential functions, where $z \in \mathbb{C}$ is a complex number.
 - (b) Given a function $\varphi \colon \mathbb{R} \to \mathbb{C}$ defined as $\varphi(x) = \exp(ix)$. We can decompose $\varphi = i_f \circ \tilde{\varphi} \circ \pi_{\sim}$, where π_{\sim} is surjective, i_{φ} is injective, and $\tilde{\varphi}$ is bijective, which can be expressed as follows:





Find *X* and *Y* in the above commutative diagram.

Hint: Consider π_{\sim} as a projection to an equivalent class, $\tilde{\varphi}$ as a modification of φ , and i_{φ} as a map from the image to the co-domain.

5. Let a third order differential equation be as follows:

$$\ell[y(t)] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

Let $\ell[y(t)] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell[y(t)]$ is non-trivial.

- (b) Find the particular solution to $\ell[y(t)] = \sin t$.
- (c) Find the particular solution to $\ell[y(t)] = e^{-t}$.
- (d)* Suppose that $\ell[y_1(t)] = f(t)$ and $\ell[y_2(t)] = g(t)$ where f(t) and g(t) are "good" functions. Find an expression to $y_3(t)$ such that $\ell[y_3(t)] = f(t) + g(t)$.
- 6. Show the following Laplace transformation by definition.

(a)
$$\mathcal{L}\{\sin(at)\} = \frac{a}{a^2 + s^2}.$$

(b)*
$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.$$

7. Given the following the results after Laplace transformation $F(s) = \mathcal{L}\{f(t)\}$, find each f(t) prior to the Laplace transformation.

(a)
$$F(s) = \frac{2s^2 + 4}{s^3 + 4s}.$$

(b)*
$$F(s) = \frac{s^2}{s^2 + 9} - 1.$$

8. Let $\mathbf{x} \in \mathbb{R}^2$, find the general solution of \mathbf{x} for:

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

9. Let $\mathbf{x} = (x_1, x_2)$ satisfy the following differential equation.

$$\mathbf{x}' = \begin{pmatrix} \frac{1}{42} & \frac{1}{21} \\ \frac{1}{14} & \frac{1}{21} \end{pmatrix} \cdot \mathbf{x}.$$

Hint: Think about the *geometric* interpretation of eigenvalues and eigenvectors and try to *simplify* the matrix. (*Otherwise, the computation is hard.*)



- 10.* (*Putnam* 2023.) Determine the smallest positive real number r such that there exists differentiable functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ satisfying:
 - f(0) > 0,
 - g(0) = 0,
 - $|f'(x)| \le |g(x)|$ for all x,
 - $|g'(x)| \le |f(x)|$ for all x, and
 - f(r) = 0.

You may give an answer <u>without</u> a rigorous proof, as the proof is out of scope of the course. *Hint:* Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.