Exam 2 Review Problem Set 4

Differential Equations

Summer 2025

Instructions:

The set of questions serves as PILOT practices to Exam 2 for the Summer 2025 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- Questions marked with an asterisk are more challenging.
- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.

1. Solve the following differential equations.

   (a)
   $$y'' + 4y = t^2 + 3e^t.$$

   (b)
   $$y'' + 2y' + y = \frac{e^{-x}}{x}.$$

2.* Find a full set of real solutions to the differential equation:

$$\frac{d^3y}{dx^3} = -y.$$

3. Let a differential equation of $y := y(x)$ be:

$$y''' + 3y'' + 3y' + y = 0.$$

Find the general solution the differential equation and give the Wronskian of your set of solutions.

4.* In our study of differential equations, our main focus is on *real-valued functions*. However, **Euler's theorem** bridges between real values and complex values.

   (a) Express $\sin(z)$ and $\cos(z)$ in terms of exponential functions, where $z \in \mathbb{C}$ is a complex number.

   (b) Given a function $\varphi\colon \mathbb{R} \to \mathbb{C}$ defined as $\varphi(x) = \exp(ix)$. We can decompose $\varphi = i_f \circ \tilde{\varphi} \circ \pi_\sim$, where $\pi_\sim$ is surjective, i_φ is injective, and $\tilde{\varphi}$ is bijective, which can be expressed as follows:

$$\mathbb{R} \xrightarrow[\pi_\sim]{} X \xrightarrow[\tilde{\varphi}]{\sim} Y \xrightarrow[i_\varphi]{} \mathbb{C}, \qquad \varphi: \mathbb{R} \to \mathbb{C}$$

Find X and Y in the above commutative diagram.
Hint: Consider $\pi_\sim$ as a projection to an equivalent class, $\tilde{\varphi}$ as a modification of φ, and i_φ as a map from the image to the co-domain.

5. Let a third order differential equation be as follows:

$$\ell\big[y(t)\big] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

Let $\ell\big[y(t)\big] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell\big[y(t)\big]$ is non-trivial.

(b) Find the particular solution to $\ell\big[y(t)\big] = \sin t$.

(c) Find the particular solution to $\ell\big[y(t)\big] = e^{-t}$.

(d)* Suppose that $\ell\big[y_1(t)\big] = f(t)$ and $\ell\big[y_2(t)\big] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\big[y_3(t)\big] = f(t) + g(t)$.

6. Show the following Laplace transformation by definition.

(a)
$$\mathcal{L}\big\{\sin(at)\big\} = \frac{a}{a^2 + s^2}.$$

(b)*
$$\mathcal{L}\big\{(f * g)(t)\big\} = \mathcal{L}\big\{f(t)\big\} + \mathcal{L}\big\{g(t)\big\}.$$

7. Given the following the results after Laplace transformation $F(s) = \mathcal{L}\big\{f(t)\big\}$, find each $f(t)$ prior to the Laplace transformation.

(a)
$$F(s) = \frac{2s^2 + 4}{s^3 + 4s}.$$

(b)*
$$F(s) = \frac{s^2}{s^2 + 9} - 1.$$

8. Let $\mathbf{x} \in \mathbb{R}^2$, find the general solution of $\mathbf{x}$ for:

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \cdot \mathbf{x}.$$

9. Let $\mathbf{x} = (x_1, x_2)$ satisfy the following differential equation.

$$\mathbf{x}' = \begin{pmatrix} \frac{1}{42} & \frac{1}{21} \\ \frac{1}{14} & \frac{1}{21} \end{pmatrix} \cdot \mathbf{x}.$$

Hint: Think about the *geometric* interpretation of eigenvalues and eigenvectors and try to *simplify* the matrix. (*Otherwise, the computation is hard.*)

10.* (*Putnam 2023.*) Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

- $f(0) > 0$,
- $g(0) = 0$,
- $|f'(x)| \leq |g(x)|$ for all x,
- $|g'(x)| \leq |f(x)|$ for all x, and
- $f(r) = 0$.

You may give an answer without a rigorous proof, as the proof is out of scope of the course.
Hint: Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.