Exam 3 Review Problem Set 5

Differential Equations

Summer 2025

Instructions:

The set of questions serves as PILOT practices to Exam 3 for the Summer 2025 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- Questions marked with an asterisk are more challenging.
- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.
- *This Problem Set is not cumulative. Please refer to the other problem sets.*

1. Let an initial value problem for linear system on $x_1 := x_1(t)$ and $x_2 := x_2(t)$ be defined as follows:

$$\begin{cases} x_1' = 3x_1 - 2x_2, & x_1(0) = 3, \\ x_2' = 2x_1 - 2x_2, & x_2(0) = \dfrac{1}{2}. \end{cases}$$

   (a) Solve for the *general solution* for the linear system by considering $\mathrm{x} = (x_1, x_2)$.

   (b) Transform the *general system* into a single equation of second order. Then solve the second-order equation. Eventually, convert your solution of one variable back to the *general solution* to $x_1(t)$ and $x_2(t)$.

   (c) Find the particular solution using the initial conditions, then graph the parameterized curve on a x_1x_2-plane with $t \geq 0$.

2. Solve the following initial value problem, represent your solution as a fundamental matrix:

$$\mathrm{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathrm{x}, \qquad \mathrm{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

3. Let a system of $\mathrm{x} = (x_1, x_2)$ be defined as:

$$\mathrm{x}' = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} \cdot \mathrm{x}.$$

   (a) Find the eigenvalues and eigenvectors for $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix}$.

(b) Give a full set of solutions to the differential equation. Plot some trajectory on the x_1x_2-plane.

(c)* Let A be an arbitrary square matrix. Show that A is non-invertible if and only if A has zero as an eigenvalue.
Note: Please avoid using the definition that the determinant is the product of all eigenvalues. Moreover, consider the geometric implication of eigenvalue to account for invertibility.

4. Let M be a square matrix, M is defined to be *nilpotent* if $M^k = 0$ for some positive integer k.

(a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

(b) If all the entries in N are rational, show that $\exp(N)$ has rational entries.

(c)* Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
Hint: Use the differences of squares for matrices.

5. Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) Show that $M^\intercal = M^{-1}$.

(b)* Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that $M^n = \mathrm{Id}_2$. Here, n is called the *order* of M.
Hint: Consider the rotational matrix geometrically, rather than arithmetically.

(c)* Let $\theta = \pi/2$, calculate the matrix exponential $\exp(M)$.
Hint: Consider the *order* of M and the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$.

6. Let a system of non-linear differential equations be defined as follows:

$$\begin{cases} x' = x - y^2, \\ y' = x + x^2 - 2y. \end{cases}$$

Find all equilibrium(s) and classify their stability locally.

7. Let a system of (x, y) be functions of variable t, and they have the following relationship:

$$x' = (1 + x)\sin y \text{ and } y' = 1 - x - \cos y.$$

Interpret the stability/type for the equilibrium at $(0, 0)$ by showing it is locally linear.

8. Suppose D and R are two parties on a non-existing country on the center of Mars. For the simplicity of this problem, they, *unfortunately*, have no elections. Therefore, we can model the amount of the

supporter for each party (in millions), denoted x_D and x_R with the following relationship:

$$\begin{cases} \dfrac{dx_D}{dt} = x_D(1 - x_D - x_R), \\ \dfrac{dx_R}{dt} = x_R(3 - 2x_D - 4x_R). \end{cases}$$

Find all possible endings (say arbitrarily long after, that is $t \to \infty$) of the number of supporters (in millions) for the two parties.

9. Suppose the tariff system in Mars (between all countries there) is based on the same formula, which is as follows:

$$\Delta\tau_i = \frac{x_i - m_i}{\varepsilon \times \varphi \times m_i}.$$

Here, $\Delta\tau_i$ means the change in tariff, x_i means the total import sale into your country from country i, m_i means the total export sale from your country to the country i, and $\varepsilon \times \varphi$ is 2.
Furthermore, a numerical estimation method in ODEs is called *Euler's Method*, and we will use the reverse of that to obtain an ODE model that:

$$\frac{d\tau(t)}{dt} \approxeq \frac{x(t) - m(t)}{2m(t)}.$$

Now, suppose there is another county, and you want to analyze the trends of tariffs with that country. With ϑ denoting their country's tariff on your country's import, we can create a system.

$$\begin{cases} \dfrac{d\tau(t)}{dt} = \dfrac{x(t) - m(t)}{2m(t)}, \\ \dfrac{d\vartheta(t)}{dt} = \dfrac{m(t) - x(t)}{2x(t)}. \end{cases}$$

For the simplicity of economics, we can model the import sale and export sale as:

$$x(t) = a - b\tau(t) \qquad \text{and} \qquad m(t) = c - d\vartheta(t),$$

where a, b, c, d are positive real constants.

(a) Write down the system of differential equations to model the tariffs as a vector $\mathrm{x}(t) = \big(\tau(t), \vartheta(t)\big)$.

(b) Find the set of all equilibrium points on this nonlinear system.

(c)* Interpret some issues with the assumptions of this model.

10.* Let a locally linearly system be defined as:

$$\frac{d\mathrm{x}}{dt} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathrm{x} + \mathbf{f}(\mathrm{x}),$$

where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. Find the necessary condition(s) in which the equilibrium(s) have a stable *center* in linear system. Then, state the stability and type (if possible).
Hint: Consider the solution for the linear case or matrix exponential.