Exam 3 Review Problem Set 6: Solutions

Differential Equations

Summer 2025

1. Find the general solution for $y = y(t)$:

$$y' + 3y = t + e^{-2t},$$

then, describe the behavior of the solution as $t \to \infty$.

Solution:

Here, one could note that this differential equation is not separable but in the form of integrating factor problem, then we find the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t 3ds\right) = \exp(3t).$$

By multiplying both sides with $\exp(3t)$, we obtain the equation:

$$y'e^{3t} + 3ye^{3t} = te^{3t} + e^{-2t}e^{3t}.$$

Clearly, we observe that the left hand side is the derivative after product rule for ye^{3t} and the right hand side can be simplified as:

$$\frac{d}{dt}[ye^{3t}] = te^{3t} + e^{t}.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{3t} &= \int te^{3t}dt + \int e^{t}dt \\ &= \frac{te^{3t}}{3} - \int \frac{1}{3}e^{3t}dt + e^{t} + C \\ &= \frac{te^{3t}}{3} - \frac{e^{3t}}{9} + e^{t} + C. \end{aligned}$$

Eventually, we divide both sides by e^{3t} to obtain that:

$$y(t) = \boxed{\frac{t}{3} - \frac{1}{9} + e^{-2t} + Ce^{-3t}}.$$

2. Draw the phase line and determine the stability of each equilibrium for the following autonomous differential equation:

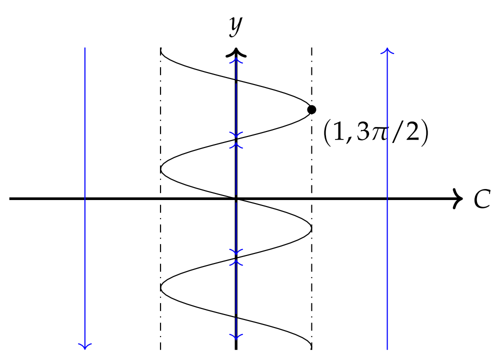
$$y' = y^2 + 2y + C, \text{ where } C \in \mathbb{R} \text{ is a constant.}$$

Determine the bifurcation values for the parameter C and sketch a bifurcation diagram.

Solution:

It is not hard to observe that $\sin y$ will intersect the axis infinitely many times, while $\sin(\mathbb{R}) = [-1, 1]$, one shall then realize that the bifurcation value would be ± 1, since when $C > 1$ or $C < -1$, there will be no equilibriums at all.

Therefore, the bifurcation diagram can be illustrated as:



3. Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:

$$y'(x) = e^{2x} + y(x) - 1.$$

Solution:
First, we write the equation in the general form:

$$\frac{dy}{dx} + (1 - e^{2x} - y) = 0.$$

Now, we take the partial derivatives to obtain that:

$$\frac{\partial}{\partial y}[1 - e^{2x} - y] = -1,$$

$$\frac{\partial}{\partial x}[1] = 0.$$

Notice that the mixed partials are not the same, the equation is $\boxed{\text{not exact}}$.
Here, we choose the integrating factor as:

$$\mu(x) = \exp\left(\int_0^x \frac{\frac{\partial}{\partial y}[1 - e^{2s} - y] - \frac{\partial}{\partial s}[1]}{1} ds\right)$$
$$= \exp\left(\int_0^x -ds\right) = \exp(-x).$$

Therefore, our equation becomes:

$$(e^{-x})\frac{dy}{dx} + (e^{-x} - e^{x} - ye^{-x}) = 0.$$

After multiplying the integrating factor, it would be exact. *We leave the repetitive check as an exercise to the readers*.
Now, we can integrate to find the solution with a $h(y)$ as function:

$$\varphi(x,y) = \int (e^{-x} - e^{x} - ye^{-x})dx = -e^{-x} - e^{x} + ye^{-x} + h(y).$$

By taking the partial derivative with respect to y, we have:

$$\partial_y \varphi(x,y) = e^{-x} + h'(y),$$

which leads to the conclusion that $h'(y) = 0$ so $h(y) = C$.
Then, we can conclude that the solution is now:

$$\varphi(x,y) = -e^{-x} - e^{x} + ye^{-x} + C = 0,$$

which is equivalently:

$$y(x) = \boxed{\widetilde{C}e^{x} + 1 + e^{2x}}.$$

4. Let a differential equation on $y := y(x)$ be defined as follows:

$$xy^2 + bx^2y + (x+y)x^2y' = 0.$$

Suppose this differential equation is exact. Find the appropriate value of b and then solve for the solution of the differential equation.

Solutions:

First, we write the differential equation as:

$$\underbrace{(xy^2 + bx^2y)}_{M(x,y)}dx + \underbrace{(x+y)x^2}_{N(x,y)}dy = 0.$$

Hence, we have the partial derivatives as:

$$\partial_y M(x,y) = 2xy + bx^2 \text{ and } \partial_x N(x,y) = 3x^2 + 2xy.$$

Hence, we have $\boxed{b=3}$ to make the differential equation exact.

Then, we have the differential equation as:

$$\underbrace{(xy^2 + 3x^2y)}_{M(x,y)}dx + \underbrace{(x+y)x^2}_{N(x,y)}dy = 0.$$

Then, we integrate M with respect as x being:

$$\phi(x,y) = \int M(x,y)dx = \frac{1}{2}x^2y^2 + x^3y + h(y).$$

When we take the derivative with respect to y being:

$$\partial_y \phi(x,y) = x^2y + x^3 + h'(y) = x^2(x+y) + h'(y),$$

hence we have $h'(y) = 0$, so $h(y) = C$, and we have the solution as:

$$\phi(x,y) = \boxed{\frac{1}{2}x^2y^2 + x^3y = C}.$$

5. Find the general solution to the following differential equations:

(a) $$y''' - 4y' = e^{-2t}.$$

(b) $$y'' + 36y = e^t \sin(6t).$$

Solution:

(a) First, we find the homogeneous case, that is $y''' - 4y' = 0$, whose characteristic equation is $r^3 - 4r = 0$, so the roots are $r = 0, 2, -2$, hence the homogeneous solution is:

$$y(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t}.$$

Given that the non-homogeneous part already exists in the equation, then our guess should be $y_p(t) = Ate^{-2t}$, which the derivatives as:

$$y_p'(t) = Ae^{-2t} - 2Ate^{-2t},$$
$$y_p''(t) = -4Ae^{-2t} + 4Ate^{-2t},$$
$$y_p'''(t) = 12Ae^{-2t} - 8Ate^{-2t}.$$

Note that when we plug into our equation, we have:

$$(12Ae^{-2t} - 8Ate^{-2t}) - 4(Ae^{-2t} - 2Ate^{-2t}) = e^{-2t}.$$

Note that the te^{-2t} term vanishes (why?), we now have:

$$8Ae^{-2t} = e^{-2t},$$

so we have that $A = 1/8$, so our general solution is:

$$y(t) = \boxed{C_1 + C_2 e^{2t} + C_3 e^{-2t} + \frac{1}{8} te^{-2t}}.$$

(b) Again, we find the homogeneous case, which is $y'' + 36y = 0$, whose characteristic equation is $r^2 + 36 = 0$, so the roots are $\pm 6i$, and the homogeneous solution is:

$$y(t) = C_1 \sin(6t) + C_2 \cos(6t).$$

Now, we need to form our guess as $y_p(t) = Ae^t \sin(6t) + Be^t \cos(6t)$, we take the derivatives as:

$$y_p'(t) = Ae^t \sin(6t) + 6Ae^t \cos(6t) + Be^t \cos(6t) - 6Be^t \sin(6t),$$
$$y_p''(t) = Ae^t \sin(6t) + 12Ae^t \cos(6t) - 36Ae^t \sin(6t) + Be^t \cos(6t) - 12Be^t \sin(6t) - 36Be^t \cos(6t)$$
$$= (-35A - 12B)e^t \sin(6t) + (12A - 35B)e^t \cos(6t).$$

When plugged back into the differential equation, we have:

$$(-35A - 12B + 36A)e^t \sin(6t) + (12A - 35B + 36B)e^t \cos(6t) = e^t \sin(6t).$$

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Then, we have a system of linear equations as:

$$\begin{cases} A - 12B = 1, \\ 12A + B = 0. \end{cases}$$

This solves into $A = \frac{1}{145}$ and $B = -\frac{12}{145}$, so the general solution is:

$$y(t) = \boxed{C_1 \sin(6t) + C_2 \cos(6t) + \frac{1}{145} e^t \sin(6t) - \frac{12}{145} e^t \cos(6t)}.$$

At this moment, we highly encourage readers to consider why the particular guess did **not** have an additional order, and why both sine and cosine are included when the derivative operators are of even orders.

6. Give the general solution to the following higher order differential equations:

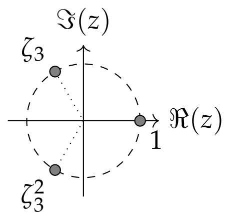
$$y^{(6)} - 2y''' + y = 0.$$

Solution:

First, we find the characteristic equation, which is a fairly easy perfect square:

$$r^6 - 2r^3 + 1 = (r^3 - 1)^2 = 0.$$

Hence, our concern follows to r being the solution to $r^3 = 1$, with double multiplicity. In particular, we have the roots being on the unit circle, with ζ_3 being the 3rd root of unity, as:



Hence, the roots of the polynomial is:

$$r = 1, \zeta_3, \zeta_3^2,$$

each with multiplicity 2, where ζ_3 and ζ_3^2 can be expressed as:

$$\zeta_3 = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2},$$

$$\zeta_3^2 = \cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}.$$

Hence, one set of solution is:

$$y_1 = e^t,$$

$$y_2 = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right),$$

$$y_3 = e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right),$$

where this set is already manipulated by Euler's identity. By multiplicity of roots:

$$y_4 = te^t,$$

$$y_5 = te^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right),$$

$$y_6 = te^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

And the set is:

$$y = \boxed{\begin{array}{l} C_1e^t + C_2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + C_3e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ C_4te^t + C_5te^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + C_6te^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{array}}.$$

7. Let a system of differential equations of $x_i(t)$ be as follows:

$$\begin{cases} x_1' = 3x_1 + 2x_2, & x_1(1) = 0, \\ x_2' = x_1 + 4x_2, & x_2(1) = 2. \end{cases}$$

(a) Solve for the solution to the initial value problem.

(b) Identify and describe the stability at equilibrium(s).

Solution:

(a) Here, we denote $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^\top$, so our system becomes:

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Here, the eigenvalues are solutions to:

$$\det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} = 0,$$

which simplifies to $\lambda^2 - 7\lambda + 10 = 0$, and further gives $\lambda_1 = 2$, $\lambda_2 = 5$. Then, we look for eigenvectors of the matrix:

- For $\lambda_1 = 2$, we have $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which gives that $\boldsymbol{\xi}_1 = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = 5$, we have $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$, which gives that $\boldsymbol{\xi}_2 = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now, the general solution must be:

$$\mathbf{x} = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t},$$

and by plugging in the initial condition, we have:

$$\begin{cases} -2C_1e^2 + C_2e^5 = 0, \\ C_1e^2 + C_2e^5 = 2. \end{cases}$$

In which the solution is $C_1 = \frac{2}{3e^2}$ and $C_2 = \frac{4}{3e^5}$, so the solution is:

$$\boxed{\begin{cases} x_1 = -\frac{4}{3}e^{2t-2} + \frac{4}{3}e^{5t-5}, \\ x_2 = \frac{2}{3}e^{2t-2} + \frac{4}{3}e^{5t-5}. \end{cases}}$$

(b) Now, we consider the equilibrium at $\mathbf{x} = \begin{pmatrix} 0 & 0 \end{pmatrix}^\top$, in which we note that both eigenvalues are positive, meaning that this is an $\boxed{\text{unstable node}}$.

8. Let $\mathrm{Id} \in \mathcal{L}(\mathbb{R}^n)$ be the identity map in an n-dimensional Euclidean space, show that the following equality holds for matrix exponential:

$$\exp(\mathrm{Id}) = e \cdot \mathrm{Id}\,.$$

Hint: Consider the matrix exponential and the Taylor expansion of $\exp(x)$.

Solution:

Proof. Here, we first note that, by definition:

$$\mathrm{Id}^k = \mathrm{Id} \ \text{ for all } k \in \mathbb{N},$$

thus, we want to expand the matrix exponential as follows:

$$\begin{aligned}\exp(\mathrm{Id}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{Id}^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{Id} \\ &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} \right) \mathrm{Id}\,.\end{aligned}$$

Recall that the Taylor expansion of e^x at 0 is:

$$e^x \sim \sum_{k=0}^{\infty} \frac{1}{k!} e^0 (x-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Evaluating the above equation at 1 gives that:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = e,$$

and hence, we have the matrix exponential as:

$$\exp(\mathrm{Id}) = e \cdot \mathrm{Id},$$

as desired. □

9. Let a system of differential equations be defined as follows, find its general solutions:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^3.$$

Solution:

Again, we first find the eigenvalues of the equation, *i.e.*:

$$\det \begin{pmatrix} 1-\lambda & 0 & 4 \\ 1 & 1-\lambda & 3 \\ 0 & 4 & 1-\lambda \end{pmatrix} = 0,$$

which is $(1-\lambda)^3 + 16 - 12(1-\lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda+1)^2(\lambda-5) = 0$.

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Now, we look for eigenvectors.

- For $\lambda_1 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $x \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$.
- For $\lambda_2 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} 4x \\ x+1 \\ -2x-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
- For $\lambda_3 = 5$, we have $\begin{pmatrix} -4 & 0 & 4 \\ 1 & -4 & 3 \\ 0 & 4 & -4 \end{pmatrix} \boldsymbol{\xi}_3 = \mathbf{0}$, which is $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + C_2 \left(t e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) + C_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}.$$

10. Determine the periodic solution, if there are any, of the following system:

$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$

Solution:
Here, we recall the formula converting between polar coordinates and Cartesian coordinates:

$$\begin{cases} x = r\cos\theta, & y = r\sin\theta, \\ rr' = xx' + yy', & r^2\theta' = xy' - yx'. \end{cases}$$

Now, we are able to convert the system as:

$$\begin{cases} rr' = x\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] + y\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right], \\ r^2\theta' = x\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] - y\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right]. \end{cases}$$

Here, by simple deductions, we trivially have:

$$rr' = \frac{x^2+y^2}{\sqrt{x^2+y^2}}(x^2+y^2-2) = \frac{r^2}{r}(r^2-2) \rightsquigarrow r' = r^2 - 2.$$
$$r^2\theta' = -x^2 - y^2 = -r^2 \rightsquigarrow \theta' = -1.$$

Thereby, we consider the radius as:

$$r' = r^2 - 2 = (r - \sqrt{2})(r + \sqrt{2}).$$

Hence, we note that the critical point is $\boxed{r = \sqrt{2}}$ (since r must be positive). Note that $r' < 0$ for $0 < r < \sqrt{2}$ and $r' > 0$ for $r > \sqrt{2}$. Hence, this is an unstable limit cycle.