# Exam 3 Review Problem Set 6

Differential Equations

Summer 2025

**Instructions:**

The set of questions serves as PILOT practices to Exam 3 for the Summer 2025 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.

1. Find the general solution for $y = y(t)$:

$$y' + 3y = t + e^{-2t},$$

then, describe the behavior of the solution as $t \to \infty$.

2. Draw the phase line and determine the stability of each equilibrium for the following autonomous differential equation:

$$y' = y^2 + 2y + C, \text{ where } C \in \mathbb{R} \text{ is a constant.}$$

Determine the bifurcation values for the parameter $C$ and sketch a bifurcation diagram.

3. Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:

$$y'(x) = e^{2x} + y(x) - 1.$$

4. Let a differential equation on $y := y(x)$ be defined as follows:

$$xy^2 + bx^2y + (x + y)x^2y' = 0.$$

Suppose this differential equation is exact. Find the appropriate value of $b$ and then solve for the solution of the differential equation.

5. Find the general solution to the following differential equations:

(a)
$$y''' - 4y' = e^{-2t}.$$
(b)
$$y'' + 36y = e^t \sin(6t).$$

6. Give the general solution to the following higher order differential equations:
$$y^{(6)} - 2y''' + y = 0.$$

7. Let a system of differential equations of $x_i(t)$ be as follows:
$$\begin{cases} x_1' = 3x_1 + 2x_2, & x_1(1) = 0, \\ x_2' = x_1 + 4x_2, & x_2(1) = 2. \end{cases}$$
   (a) Solve for the solution to the initial value problem.
   (b) Identify and describe the stability at equilibrium(s).

8. Let $\mathrm{Id} \in \mathcal{L}(\mathbb{R}^n)$ be the identity map in an $n$-dimensional Euclidean space, show that the following equality holds for matrix exponential:
$$\exp(\mathrm{Id}) = e \cdot \mathrm{Id}\,.$$
*Hint:* Consider the matrix exponential and the Taylor expansion of $\exp(x)$.

9. Let a system of differential equations be defined as follows, find its general solutions:
$$\mathrm{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathrm{x}, \qquad \mathrm{x} \in \mathbb{R}^3.$$

10. Determine the periodic solution, if there are any, of the following system:
$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$