PILOT Exam 3 Review

Differential Equations

Johns Hopkins University

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As you prepare for exam 3, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/SU25/`
  - Find the review problem sets for exam 3.
    - Also, review the previous problem sets for the first two exams. The last exam is cumulative.
  - Consult the archives page for PILOT sets from the semester.
- Review the *homework/quiz sets* provided by the instructor.
- Join the PILOT Exam 3 Review Session. (You are here.)

Plan for today:

1. Go over all contents that we have covered for this semester so far.
2. In the end, we will open the poll to you. Please indicate which problems from the Review Set that you want us to go over.

Part 1:
Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

System of First Order Linear ODEs (Continued)

- Repeated Eigenvalues
  - Algebraic Multiplicity and Geometric Multiplicity
- Phase Portraits
  - Node Graph
  - Spiral/Center Graph
  - Repeated Eigenvalue Graph
- Fundamental Matrix
- Non-homogeneous Cases
  - Diagonalization
  - Undetermined Coefficients
  - Variation of Parameters

For repeated eigenvalue r with only one (linearly independent) eigenvector, if a given a solution is $\mathbf{x}^{(1)} = \boldsymbol{\xi} e^{rt}$, the other solution would be:

$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt},$$

where $(A - Ir) \cdot \boldsymbol{\eta} = \boldsymbol{\xi}$, and $\mathbf{x}^{(2)}$ is called the *generalized eigenvector*.

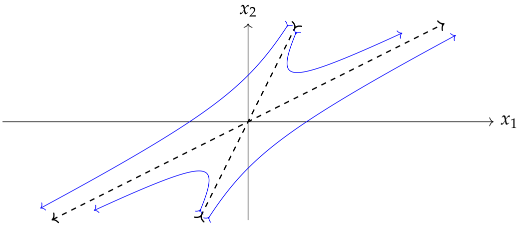
Algebraic Multiplicity and Geometric Multiplicity

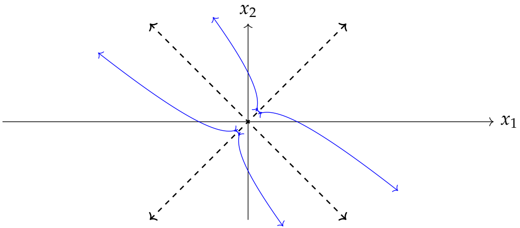
The algebraic multiplicity refers to the multiplicity of root in the characteristic polynomial, and the geometric multiplicity refers to the dimension of the eigenspace associated with the eigenvalue.

- The algebraic multiplicity will be no less than the geometric multiplicity for each eigenvalue.
- We need the generalized eigenvector when the algebraic multiplicity is larger than the geometric multiplicity.

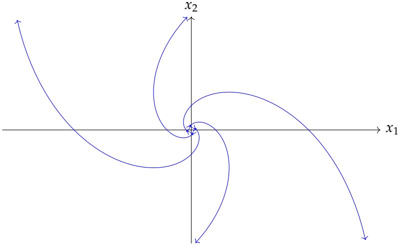
In particular, we can sketch the linear system of $\mathbb{R}^2$ in terms of phase portraits given the eigenvalues and eigenvectors.

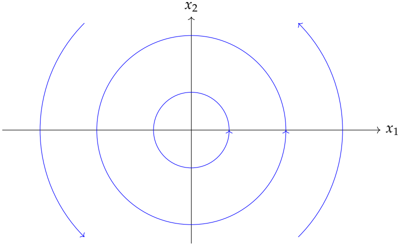
- For a node graph, we have it as (directions might vary):



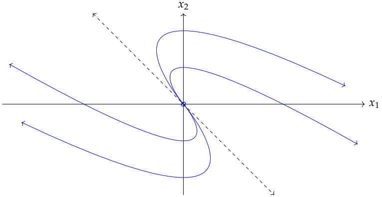


- For a spiral/center graph, we have it as (directions might vary):

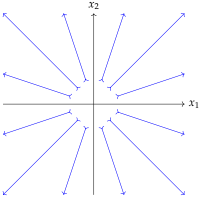




- For repeated eigenvalues with less geometric multiplicity, the solution is (directions might vary):



- If the geometric multiplicity is the same, the graph is simply a radial shape (directions might vary):



The exponential of Matrix is defined to be:

$$\exp(tA) = I + \sum_{n=1}^{\infty} \frac{(tA)^n}{n!},$$

where A^n is the result of n square matrices of A multiplying themselves.

Using Matrix Exponential to Solve Linear Systems

The special case of fundamental matrix is defined to be Φ where:

$$\begin{cases} \Phi' = A \cdot \Phi, \\ \Phi(t_0) = I, \end{cases}$$

so that the fundamental matrix Φ can be calculated by:

$$\Phi(t) = \exp(tA) \cdot \big(\exp(t_0 A)\big)^{-1}.$$

Let the differential equation be:

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{g}(t),$$

where $\mathbf{g}(t)$ is a smooth vector-valued function. Let ϕ be its fundamental matrix, then the non-homogeneous cases can be solved by the following approaches.

Diagonalization

Utilizes T as the matrix of eigenvectors and D as the diagonal matrix of eigenvalues. Accordingly, let $\mathbf{x} = T\mathbf{y}$.
Then, $\mathbf{x}' = T\mathbf{y}' = AT\mathbf{y} + \mathbf{g} = TD\mathbf{y} + \mathbf{g}$, which means that $\mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g}$ and the differential equation is now degenerated.

Undetermined Coefficients

Same as in single equations, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $\mathbf{g}(t)$. Some brief strategies are:

Non-homogeneous Comp. in $\mathbf{g}(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} \mathbf{a_i} t^i$	$\sum_{i=0}^{d} \mathbf{c_i} t^i$
Trig.:	$\mathbf{a_1} \sin(b_1 t)$ and $\mathbf{a_2} \cos(b_2 t)$	$\mathbf{c_1} \sin(b_1 x) + \mathbf{c_2} \cos(b_2 x)$
Exp.:	$\mathbf{a} e^{bt}$	$\mathbf{c} e^{bt}$

Again, the guesses are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

Variation of Parameters

We utilizes that:

$$\Psi \cdot \mathbf{u}' = \mathbf{g},$$

where this equation can be solved by:

$$u_i' = \frac{W_i}{\det(\Psi)},$$

where W_i is defined by the Wronskian of the matrix replacing the i-th column with $\mathbf{g}(t)$.
There, the particular solution is:

$$\mathbf{x}_p = \Psi \cdot \mathbf{u}.$$

Non-linear Systems

- Linear Approximation
  - Autonomous Systems
- Stability
- Limit Cycles

For non-linear system $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}'$, if $F, G \in C^2$ and the system is locally linear, the approximation at critical point (x_0, y_0) is:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \mathbf{J}(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where Jacobian is:

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}.$$

Autonomous Systems

When $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} F(y) \\ G(x) \end{pmatrix}$, it can be solved implicitly for:

$$\frac{dy}{dx} = \frac{G(x)}{F(y)}.$$

For linearized system with eigenvalues r_1, r_2, the stability can be concluded as follows:

Eigenvalues	**Linear System**		**Nonlinear System**	
	Type	**Stability**	**Type**	**Stability**
Eigenvalues are λ_1 and λ_2				
$0 < \lambda_1 < \lambda_2$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable	Node or Spiral Points	Asymptotically Stable

Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$				
$\alpha > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$\alpha = 0$	Center	Stable	Center or Spiral Point	Indeterminate
$\alpha < 0$	Spiral Point	Asymptotically Stable	Spiral Point	Asymptotically Stable

A closed trajectory or periodic solution repeats back to itself with period τ:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Closed trajectories with either side converging to/diverging from the solution is a limit cycle.

Conversion to Polar Coordinates

A Cartesian coordinate can be converted by:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ rr' = xx' + yy', \\ r^2\theta' = xy' - yx'. \end{cases}$$

For a linear system $x = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$ with $F, G \in C^1$:

1. A closed trajectory of the system must enclose at least 1 critical point.
2. If it only encloses 1 critical point, then that critical point cannot be saddle point.
3. If there are no critical points, there are no closed trajectories.
4. If the unique critical point is saddle, there are no trajectories.
5. For a simple connected domain D in the xy-plane with no holes. If $F_x + G_y$ had the same sign throughout D, then there is no closed trajectories in D.

Part 2:
Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through selected problems from the practice problem set.
- We are also open to conceptual questions with the course.

Good luck on your final exam.

Thank you for being with PILOT for Differential Equations over this semester.

We wish the best of all for you future academic and career pursuits!